Operations Research: An Introduction

Eighth Edition

Hamdy A. Taha
University of Arkansas, Fayetteville

PEARSON
Prentice Hall

Upper Saddle River, New Jersey 07458

Library of Congress Cataloging-in-Publication Data

Taha, Hamdy A.
Operations research: an introduction / Hamdy A. Taha.—8th ed.
p. cm.
Includes bibliographical references and index.
ISBN 0-13-188923-0
1. Operations research. 2. Programming (Mathematics) 1. Title.

T57.6.T3 1997 96-37160
003 -dc21

96-37160

Vice President and Editorial Director, ECS: *Marcia J. Horton*
Senior Editor: *Holly Stark*
Executive Managing Editor: *Vince O'Brien*
Managing Editor: *David A. George*
Production Editor: *Craig Little*
Director of Creative Services: *Paul Belfanti*
Art Director: *Jayne Conte*
Cover Designer: *Bruce Kenselaar*
Art Editor: *Greg Dulles*
Manufacturing Manager: *Alexis Heydt-Long*
Manufacturing Buyer: *Lisa McDowell*

Pearson Prentice Hall
Pearson Education, Inc.
Upper Saddle River, NJ 07458

Printed in the United States of America
10 9 8 7 6 5 4 3 2

ISBN 0-13-188923-0

Pearson Education Ltd., *London*
Pearson Education Australia Pty. Ltd., *Sydney*
Pearson Education Singapore, Pte. Ltd.
Pearson Education North Asia Ltd., *Hong Kong*
Pearson Education Canada, Inc., *Toronto*
Pearson Educación de Mexico, S.A. de C.V.
Pearson Education—Japan, *Tokyo*
Pearson Education Malaysia, Pte. Ltd.
Pearson Education, Inc., *Upper Saddle River, New Jersey*

To Karen

Los ríos no llevan agua,
el sol las fuentes secó . . .
¡Yo sé donde hay una fuente
que no ha de secar el sol!
La fuente que no se agota
es mi propio corazón . . .

—V. Ruiz Aguilera (1862)

Contents

On the CD-ROM

Preface

The eighth edition is a major revision that streamlines the presentation of the text material with emphasis on the applications and computations in operations research:

- Chapter 2 is dedicated to linear programming modeling, with applications in the areas of urban renewal, currency arbitrage, investment, production planning, blending, scheduling, and trim loss. New end-of-section problems deal with topics ranging from water quality management and traffic control to warfare.
- Chapter 3 presents the general LP sensitivity analysis, including dual prices and reduced costs, in a simple and straightforward manner as a direct extension of the simplex tableau computations.
- Chapter 4 is now dedicated to LP post-optimal analysis based on duality.
- An Excel-based combined nearest neighbor-reversal heuristic is presented for the traveling salesperson problem.
- Markov chains treatment has been expanded into new Chapter 17.
- The totally new Chapter 24* presents 15 fully developed real-life applications. The analysis, which often cuts across more than one OR technique (e.g., heuristics and LP, or ILP and queuing), deals with the modeling, data collection, and computational aspects of solving the problem. These applications are cross-referenced in pertinent chapters to provide an appreciation of the use of OR techniques in practice.
- The new Appendix E* includes approximately 50 mini cases of real-life situations categorized by chapters.
- More than 1000 end-of-section problem are included in the book.
- Each chapter starts with a *study guide* that facilitates the understanding of the material and the effective use of the accompanying software.
- The integration of software in the text allows testing concepts that otherwise could not be presented effectively:
    1. Excel spreadsheet implementations are used throughout the book, including dynamic programming, traveling salesperson, inventory, AHP, Bayes' probabilities, "electronic" statistical tables, queuing, simulation, Markov chains, and nonlinear programming. The interactive user input in some spreadsheets promotes a better understanding of the underlying techniques.
    2. The use of Excel Solver has been expanded throughout the book, particularly in the areas of linear, network, integer, and nonlinear programming.
    3. The powerful commercial modeling language, AMPL®, has been integrated in the book using numerous examples ranging from linear and network to

*Contained on the CD-ROM.

integer and nonlinear programming. The syntax of AMPL is given in Appendix A and its material cross-referenced within the examples in the book.
4. TORA continue to play the key role of tutorial software.

- All computer-related material has been deliberately compartmentalized either in separate sections or as subsection titled *AMPL/Excel/Solver/TORA moment* to minimize disruptions in the main presentation in the book.

To keep the page count at a manageable level, some sections, complete chapters, and two appendixes have been moved to the accompanying CD. The selection of the excised material is based on the author's judgment regarding frequency of use in introductory OR classes.

ACKNOWLEDGMENTS

I wish to acknowledge the importance of the seventh edition reviews provided by Layek L. Abdel-Malek, New Jersey Institute of Technology, Evangelos Triantaphyllou, Louisiana State University, Michael Branson, Oklahoma State University, Charles H. Reilly, University of Central Florida, and Mazen Arafeh, Virginia Polytechnic Institute and State University. In particular, I owe special thanks to two individuals who have influenced my thinking during the preparation of the eighth edition: R. Michael Harnett (Kansas State University), who over the years has provided me with valuable feedback regarding the organization and the contents of the book, and Richard H. Bernhard (North Carolina State University), whose detailed critique of the seventh edition prompted a reorganization of the opening chapters in this edition.

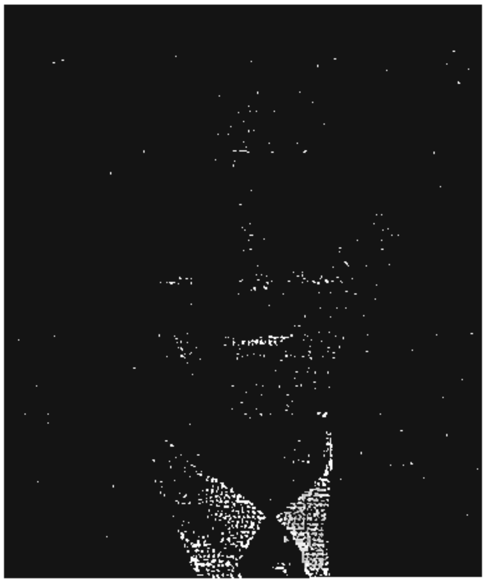
Robert Fourer (Northwestern University) patiently provided me with valuable feedback about the AMPL material presented in this edition. I appreciate his help in editing the material and for suggesting changes that made the presentation more readable. I also wish to acknowledge his help in securing permissions to include the AMPL student version and the solvers CPLEX, KNITRO, LPSOLVE, LOQO, and MINOS on the accompanying CD.

As always, I remain indebted to my colleagues and to hundreds of students for their comments and their encouragement. In particular, I wish to acknowledge the support I receive from Yuh-Wen Chen (Da-Yeh University, Taiwan), Miguel Crispin (University of Texas, El Paso), David Elizandro (Tennessee Tech University), Rafael Gutiérrez (University of Texas, El Paso), Yasser Hosni (University of Central Florida), Erhan Kutanoglu (University of Texas, Austin), Robert E. Lewis (United States Army Management Engineering College), Gino Lim (University of Houston), Scott Mason (University of Arkansas), Heather Nachtman (University of Arkanas), Manuel Rossetti (University of Arkansas), Tarek Taha (JB Hunt, Inc.), and Nabeel Yousef (University of Central Florida).

I wish to express my sincere appreciation to the Pearson Prentice Hall editorial and production teams for their superb assistance during the production of the book: Dee Bernhard (Associate Editor), David George (Production Manager - Engineering), Bob Lentz (Copy Editor), Craig Little (Production Editor), and Holly Stark (Senior Acquisitions Editor).

HAMDY A. TAHA
hat@uark.edu
http://ineg.uark.edu/TahaORbook/

About the Author



Hamdy A. Taha is a University Professor Emeritus of Industrial Engineering with the University of Arkansas, where he taught and conducted research in operations research and simulation. He is the author of three other books on integer programming and simulation, and his works have been translated into Malay, Chinese, Korean, Spanish, Japanese, Russian, Turkish, and Indonesian. He is also the author of several book chapters, and his technical articles have appeared in *European Journal of Operations Research, IEEE Transactions on Reliability, IIE Transactions, Interfaces, Management Science, Naval Research Logistics Quarterly, Operations Research,* and *Simulation*.

Professor Taha was the recipient of the Alumni Award for excellence in research and the university-wide Nadine Baum Award for excellence in teaching, both from the University of Arkansas, and numerous other research and teaching awards from the College of Engineering, University of Arkansas. He was also named a Senior Fulbright Scholar to Carlos III University, Madrid, Spain. He is fluent in three languages and has held teaching and consulting positions in Europe, Mexico, and the Middle East.

Trademarks

AMPL is a registered trademark of AMPL Optimization, LLC, 900 Sierra Pl. SE, Albuquerque, NM 87108-3379.

CPLEX is a registered trademark of ILOG, Inc., 1080 Linda Vista Ave., Mountain View, CA 94043.

KNITRO is a registered trademark of Ziena Optimization Inc., 1801 Maple Ave, Evanston, IL 60201.

LOQO is a trademark of Princeton University, Princeton, NJ 08544.

Microsoft, Windows, and Excel registered trademarks of Microsoft Corporation in the United States and/or other countries.

MINOS is a trademark of Stanford University, Stanford, CA 94305.

Solver is a trademark of Frontline Systems, Inc., 7617 Little River Turnpike, Suite 960, Annandale, VA 22003.

TORA is a trademark of SimTec, Inc., P.O. Box 3492, Fayetteville, AR 72702

Note: Other product and company names that are mentioned herein may be trademarks or registered trademarks of their respective owners in the United States and/or other countries.

1.1

CHAPTER 1

What Is Operations Research?

Chapter Guide. The first formal activities of Operations Research (OR) were initiated in England during World War II, when a team of British scientists set out to make scientifically based decisions regarding the best utilization of war materiel. After the war, the ideas advanced in military operations were adapted to improve efficiency and productivity in the civilian sector.

This chapter will familiarize you with the basic terminology of operations research, including mathematical modeling, feasible solutions, optimization, and iterative computations. You will learn that defining the problem correctly is the most important (and most difficult) phase of practicing OR. The chapter also emphasizes that, while mathematical modeling is a cornerstone of OR, intangible (unquantifiable) factors (such as human behavior) must be accounted for in the final decision. As you proceed through the book, you will be presented with a variety of applications through solved examples and chapter problems. In particular, Chapter 24 (on the CD) is entirely devoted to the presentation of fully developed case analyses. Chapter materials are cross-referenced with the cases to provide an appreciation of the use of OR in practice.

1.1 OPERATIONS RESEARCH MODELS

Imagine that you have a 5-week business commitment between Fayetteville (FYV) and Denver (DEN). You fly out of Fayetteville on Mondays and return on Wednesdays. A regular round-trip ticket costs $400, but a 20% discount is granted if the dates of the ticket span a weekend. A one-way ticket in either direction costs 75% of the regular price. How should you buy the tickets for the 5-week period?

We can look at the situation as a decision-making problem whose solution requires answering three questions:

1. What are the decision **alternatives?**
2. Under what **restrictions** is the decision made?
3. What is an appropriate **objective criterion** for evaluating the alternatives?

Three alternatives are considered:

1. Buy five regular FYV-DEN-FYV for departure on Monday and return on Wednesday of the same week.
2. Buy one FYV-DEN, four DEN-FYV-DEN that span weekends, and one DEN-FYV.
3. Buy one FYV-DEN-FYV to cover Monday of the first week and Wednesday of the last week and four DEN-FYV-DEN to cover the remaining legs. All tickets in this alternative span at least one weekend.

The restriction on these options is that you should be able to leave FYV on Monday and return on Wednesday of the same week.

An obvious objective criterion for evaluating the proposed alternative is the price of the tickets. The alternative that yields the smallest cost is the best. Specifically, we have

$$\text{Alternative 1 cost} = 5 \times 400 = \$2000$$

$$\text{Alternative 2 cost} = .75 \times 400 + 4 \times (.8 \times 400) + .75 \times 400 = \$1880$$

$$\text{Alternative 3 cost} = 5 \times (.8 \times 400) = \mathbf{\$1600}$$

Thus, you should choose alternative 3.

Though the preceding example illustrates the three main components of an OR model—alternatives, objective criterion, and constraints—situations differ in the details of how each component is developed and constructed. To illustrate this point, consider forming a maximum-area rectangle out of a piece of wire of length L inches. What should be the width and height of the rectangle?

In contrast with the tickets example, the number of alternatives in the present example is not finite; namely, the width and height of the rectangle can assume an infinite number of values. To formalize this observation, the alternatives of the problem are identified by defining the width and height as continuous (algebraic) variables.

Let

$$w = \text{width of the rectangle in inches}$$

$$h = \text{height of the rectangle in inches}$$

Based on these definitions, the restrictions of the situation can be expressed verbally as

1. Width of rectangle + Height of rectangle = Half the length of the wire
2. Width and height cannot be negative

These restrictions are translated algebraically as

1. $2(w + h) = L$
2. $w \geq 0, h \geq 0$

The only remaining component now is the objective of the problem; namely, maximization of the area of the rectangle. Let z be the area of the rectangle, then the complete model becomes

$$\text{Maximize } z = wh$$

subject to

$$2(w + h) = L$$
$$w, h \geq 0$$

The optimal solution of this model is $w = h = \frac{L}{4}$, which calls for constructing a square shape.

Based on the preceding two examples, the general OR model can be organized in the following general format:

Maximize or minimize **Objective Function**
subject to
Constraints

A solution of the mode is **feasible** if it satisfies all the constraints. It is **optimal** if, in addition to being feasible, it yields the best (maximum or minimum) value of the objective function. In the tickets example, the problem presents three feasible alternatives, with the third alternative yielding the optimal solution. In the rectangle problem, a feasible alternative must satisfy the condition $w + h = \frac{L}{2}$ with w and h assuming nonnegative values. This leads to an infinite number of feasible solutions and, unlike the tickets problem, the optimum solution is determined by an appropriate mathematical tool (in this case, differential calculus).

Though OR models are designed to "optimize" a specific objective criterion subject to a set of constraints, the quality of the resulting solution depends on the completeness of the model in representing the real system. Take, for example, the tickets model. If one is not able to identify all the dominant alternatives for purchasing the tickets, then the resulting solution is optimum only relative to the choices represented in the model. To be specific, if alternative 3 is left out of the model, then the resulting "optimum" solution would call for purchasing the tickets for $1880, which is a **suboptimal** solution. The conclusion is that "the" optimum solution of a model is best only for *that* model. If the model happens to represent the real system reasonably well, then its solution is optimum also for the real situation.

PROBLEM SET 1.1A

1. In the tickets example, identify a fourth feasible alternative.
2. In the rectangle problem, identify two feasible solutions and determine which one is better.
3. Determine the optimal solution of the rectangle problem. (*Hint*: Use the constraint to express the objective function in terms of one variable, then use differential calculus.)

4. Amy, Jim, John, and Kelly are standing on the east bank of a river and wish to cross to the west side using a canoe. The canoe can hold at most two people at a time. Amy, being the most athletic, can row across the river in 1 minute. Jim, John, and Kelly would take 2, 5, and 10 minutes, respectively. If two people are in the canoe, the slower person dictates the crossing time. The objective is for all four people to be on the other side of the river in the shortest time possible.
   - (a) Identify at least two feasible plans for crossing the river (remember, the canoe is the only mode of transportation and it cannot be shuttled empty).
   - (b) Define the criterion for evaluating the alternatives.
   - *(c)[1] What is the smallest time for moving all four people to the other side of the river?

*5. In a baseball game, Jim is the pitcher and Joe is the batter. Suppose that Jim can throw either a fast or a curve ball at random. If Joe correctly predicts a curve ball, he can maintain a .500 batting average, else if Jim throws a curve ball and Joe prepares for a fast ball, his batting average is kept down to .200. On the other hand, if Joe correctly predicts a fast ball, he gets a .300 batting average, else his batting average is only .100.
   - (a) Define the alternatives for this situation.
   - (b) Define the objective function for the problem and discuss how it differs from the familiar optimization (maximization or minimization) of a criterion.

6. During the construction of a house, six joists of 24 feet each must be trimmed to the correct length of 23 feet. The operations for cutting a joist involve the following sequence:

Operation	Time (seconds)
1. Place joist on saw horses	15
2. Measure correct length (23 feet)	5
3. Mark cutting line for circular saw	5
4. Trim joist to correct length	20
5. Stack trimmed joist in a designated area	20

Three persons are involved: Two loaders must work simultaneously on operations 1, 2, and 5, and one cutter handles operations 3 and 4. There are two pairs of saw horses on which untrimmed joists are placed in preparation for cutting, and each pair can hold up to three side-by-side joists. Suggest a good schedule for trimming the six joists.

1.2 SOLVING THE OR MODEL

In OR, we do not have a single general technique to solve all mathematical models that can arise in practice. Instead, the type and complexity of the mathematical model dictate the nature of the solution method. For example, in Section 1.1 the solution of the tickets problem requires simple ranking of alternatives based on the total purchasing price, whereas the solution of the rectangle problem utilizes differential calculus to determine the maximum area.

The most prominent OR technique is **linear programming**. It is designed for models with linear objective and constraint functions. Other techniques include **integer programming** (in which the variables assume integer values), **dynamic programming**

[1] An asterisk (*) designates problems whose solution is provided in Appendix C.

(in which the original model can be decomposed into more manageable subproblems), **network programming** (in which the problem can be modeled as a network), and **nonlinear programming** (in which functions of the model are nonlinear). These are only a few among many available OR tools.

A peculiarity of most OR techniques is that solutions are not generally obtained in (formulalike) closed forms. Instead, they are determined by **algorithms.** An algorithm provides fixed computational rules that are applied repetitively to the problem, with each repetition (called **iteration**) moving the solution closer to the optimum. Because the computations associated with each iteration are typically tedious and voluminous, it is imperative that these algorithms be executed on the computer.

Some mathematical models may be so complex that it is impossible to solve them by any of the available optimization algorithms. In such cases, it may be necessary to abandon the search for the *optimal* solution and simply seek a *good* solution using **heuristics** or *rules of thumb.*

1.3 QUEUING AND SIMULATION MODELS

Queuing and simulation deal with the study of waiting lines. They are not optimization techniques; rather, they determine measures of performance of the waiting lines, such as average waiting time in queue, average waiting time for service, and utilization of service facilities.

Queuing models utilize probability and stochastic models to analyze waiting lines, and simulation estimates the measures of performance by imitating the behavior of the real system. In a way, simulation may be regarded as the next best thing to observing a real system. The main difference between queuing and simulation is that queuing models are purely mathematical, and hence are subject to specific assumptions that limit their scope of application. Simulation, on the other hand, is flexible and can be used to analyze practically any queuing situation.

The use of simulation is not without drawbacks. The process of developing simulation models is costly in both time and resources. Moreover, the execution of simulation models, even on the fastest computer, is usually slow.

1.4 ART OF MODELING

The illustrative models developed in Section 1.1 are true representations of real situations. This is a rare occurrence in OR, as the majority of applications usually involve (varying degrees of) approximations. Figure 1.1 depicts the levels of abstraction that characterize the development of an OR model. We abstract the assumed real world from the real situation by concentrating on the dominant variables that control the behavior of the real system. The model expresses in an amenable manner the mathematical functions that represent the behavior of the assumed real world.

To illustrate levels of abstraction in modeling, consider the Tyko Manufacturing Company, where a variety of plastic containers are produced. When a production order is issued to the production department, necessary raw materials are acquired from the company's stocks or purchased from outside sources. Once the production batch is completed, the sales department takes charge of distributing the product to customers.

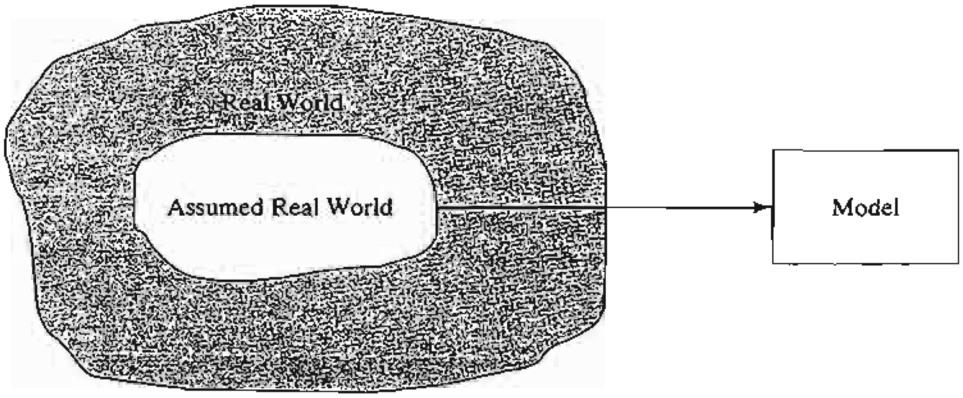


FIGURE 1.1
Levels of abstraction in model development

A logical question in the analysis of Tyko's situation is the determination of the size of a production batch. How can this situation be represented by a model?

Looking at the overall system, a number of variables can bear directly on the level of production, including the following (partial) list categorized by departments.

1. *Production Department:* Production capacity expressed in terms of available machine and labor hours, in-process inventory, and quality control standards.
2. *Materials Department:* Available stock of raw materials, delivery schedules from outside sources, and storage limitations.
3. *Sales Department:* Sales forecast, capacity of distribution facilities, effectiveness of the advertising campaign, and effect of competition.

Each of these variables affects the level of production at Tyko. Trying to establish explicit functional relationships between them and the level of production is a difficult task indeed.

A first level of abstraction requires defining the boundaries of the assumed real world. With some reflection, we can approximate the real system by two dominant variables:

1. Production rate.
2. Consumption rate.

Determination of the production rate involves such variables as production capacity, quality control standards, and availability of raw materials. The consumption rate is determined from the variables associated with the sales department. In essence, simplification from the real world to the assumed real world is achieved by "lumping" several real-world variables into a single assumed-real-world variable.

It is easier now to abstract a model from the assumed real world. From the production and consumption rates, measures of excess or shortage inventory can be established. The abstracted model may then be constructed to balance the conflicting costs of excess and shortage inventory—i.e., to minimize the total cost of inventory.

1.5 MORE THAN JUST MATHEMATICS

Because of the mathematical nature of OR models, one tends to think that an OR study is *always* rooted in mathematical analysis. Though mathematical modeling is a cornerstone of OR, simpler approaches should be explored first. In some cases, a "common sense" solution may be reached through simple observations. Indeed, since the human element invariably affects most decision problems, a study of the psychology of people may be key to solving the problem. Three illustrations are presented here to support this argument.

1. Responding to complaints of slow elevator service in a large office building, the OR team initially perceived the situation as a waiting-line problem that might require the use of mathematical queuing analysis or simulation. After studying the behavior of the people voicing the complaint, the psychologist on the team suggested installing full-length mirrors at the entrance to the elevators. Miraculously the complaints disappeared, as people were kept occupied watching themselves and others while waiting for the elevator.

2. In a study of the check-in facilities at a large British airport, a United States-Canadian consulting team used queuing theory to investigate and analyze the situation. Part of the solution recommended the use of well-placed signs to urge passengers who were within 20 minutes from departure time to advance to the head of the queue and request immediate service. The solution was not successful, because the passengers, being mostly British, were "conditioned to very strict queuing behavior" and hence were reluctant to move ahead of others waiting in the queue.

3. In a steel mill, ingots were first produced from iron ore and then used in the manufacture of steel bars and beams. The manager noticed a long delay between the ingots production and their transfer to the next manufacturing phase (where end products were manufactured). Ideally, to reduce the reheating cost, manufacturing should start soon after the ingots left the furnaces. Initially the problem was perceived as a line-balancing situation, which could be resolved either by reducing the output of ingots or by increasing the capacity of the manufacturing process. The OR team used simple charts to summarize the output of the furnaces during the three shifts of the day. They discovered that, even though the third shift started at 11:00 P.M., most of the ingots were produced between 2:00 and 7:00 A.M. Further investigation revealed that third-shift operators preferred to get long periods of rest at the start of the shift and then make up for lost production during morning hours. The problem was solved by "leveling out" the production of ingots throughout the shift.

Three conclusions can be drawn from these illustrations:

1. Before embarking on sophisticated mathematical modeling, the OR team should explore the possibility of using "aggressive" ideas to resolve the situation. The solution of the elevator problem by installing mirrors is rooted in human psychology rather than in mathematical modeling. It is also simpler and less costly than any recommendation a mathematical model might have produced. Perhaps this is the reason OR teams usually include the expertise of "outsiders" from nonmathematical fields (psychology in the case of the elevator problem). This point was recognized and implemented by the first OR team in Britain during World War II.

2. Solutions are rooted in people and not in technology. Any solution that does not take human behavior into account is apt to fail. Even though the mathematical solution of the British airport problem may have been sound, the fact that the consulting team was not aware of the cultural differences between the United States and Britain (Americans and Canadians tend to be less formal) resulted in an unimplementable recommendation.

3. An OR study should never start with a bias toward using a specific mathematical tool before its use can be justified. For example, because linear programming is a successful technique, there is a tendency to use it as the tool of choice for modeling "any" situation. Such an approach usually leads to a mathematical model that is far removed from the real situation. It is thus imperative that we first analyze available data, using the simplest techniques where possible (e.g., averages, charts, and histograms), with the objective of pinpointing the source of the problem. Once the problem is defined, a decision can be made regarding the most appropriate tool for the solution.[2] In the steel mill problem, simple charting of the ingots production was all that was needed to clarify the situation.

1.6 PHASES OF AN OR STUDY

An OR study is rooted in *teamwork*, where the OR analysts and the client work side by side. The OR analysts' expertise in modeling must be complemented by the experience and cooperation of the client for whom the study is being carried out.

As a decision-making tool, OR is both a science and an art. It is a science by virtue of the mathematical techniques it embodies, and it is an art because the success of the phases leading to the solution of the mathematical model depends largely on the creativity and experience of the operations research team. Willemain (1994) advises that "effective [OR] practice requires more than analytical competence: It also requires, among other attributes, technical judgement (e.g., when and how to use a given technique) and skills in communication and organizational survival."

It is difficult to prescribe specific courses of action (similar to those dictated by the precise theory of mathematical models) for these intangible factors. We can, however, offer general guidelines for the implementation of OR in practice.

The principal phases for implementing OR in practice include

1. Definition of the problem.
2. Construction of the model.

[2]Deciding on a specific mathematical model before justifying its use is like "putting the cart before the horse," and it reminds me of the story of a frequent air traveler who was paranoid about the possibility of a terrorist bomb on board the plane. He calculated the probability that such an event could occur, and though quite small, it wasn't small enough to calm his anxieties. From then on, he always carried a bomb in his briefcase on the plane because, according to his calculations, the probability of having *two* bombs aboard the plane was practically zero!

3. Solution of the model.
4. Validation of the model.
5. Implementation of the solution.

Phase 3, dealing with *model solution*, is the best defined and generally the easiest to implement in an OR study, because it deals mostly with precise mathematical models. Implementation of the remaining phases is more an art than a theory.

Problem definition involves defining the scope of the problem under investigation. This function should be carried out by the entire OR team. The aim is to identify three principal elements of the decision problem: (1) description of the decision alternatives, (2) determination of the objective of the study, and (3) specification of the limitations under which the modeled system operates.

Model construction entails an attempt to translate the problem definition into mathematical relationships. If the resulting model fits one of the standard mathematical models, such as linear programming, we can usually reach a solution by using available algorithms. Alternatively, if the mathematical relationships are too complex to allow the determination of an analytic solution, the OR team may opt to simplify the model and use a heuristic approach, or they may consider the use of simulation, if appropriate. In some cases, mathematical, simulation, and heuristic models may be combined to solve the decision problem, as the case analyses in Chapter 24 demonstrate.

Model solution is by far the simplest of all OR phases because it entails the use of well-defined optimization algorithms. An important aspect of the model solution phase is *sensitivity analysis*. It deals with obtaining additional information about the behavior of the optimum solution when the model undergoes some parameter changes. Sensitivity analysis is particularly needed when the parameters of the model cannot be estimated accurately. In these cases, it is important to study the behavior of the optimum solution in the neighborhood of the estimated parameters.

Model validity checks whether or not the proposed model does what it purports to do—that is, does it predict adequately the behavior of the system under study? Initially, the OR team should be convinced that the model's output does not include "surprises." In other words, does the solution make sense? Are the results intuitively acceptable? On the formal side, a common method for checking the validity of a model is to compare its output with historical output data. The model is valid if, under similar input conditions, it reasonably duplicates past performance. Generally, however, there is no assurance that future performance will continue to duplicate past behavior. Also, because the model is usually based on careful examination of past data, the proposed comparison is usually favorable. If the proposed model represents a new (nonexisting) system, no historical data would be available. In such cases, we may use simulation as an independent tool for verifying the output of the mathematical model.

Implementation of the solution of a validated model involves the translation of the results into understandable operating instructions to be issued to the people who will administer the recommended system. The burden of this task lies primarily with the OR team.

1.7 ABOUT THIS BOOK

Morris (1967) states that "the teaching of models is not equivalent to the teaching of modeling." I have taken note of this important statement during the preparation of the eighth edition, making an effort to introduce the art of modeling in OR by including realistic models throughout the book. Because of the importance of computations in OR, the book presents extensive tools for carrying out this task, ranging from the tutorial aid TORA to the commercial packages Excel, Excel Solver, and AMPL.

A first course in OR should give the student a good foundation in the mathematics of OR as well as an appreciation of its potential applications. This will provide OR users with the kind of confidence that normally would be missing if training were concentrated only on the philosophical and artistic aspects of OR. Once the mathematical foundation has been established, you can increase your capabilities in the artistic side of OR modeling by studying published practical cases. To assist you in this regard, Chapter 24 includes 15 fully developed and analyzed cases that cover most of the OR models presented in this book. There are also some 50 cases that are based on real-life applications in Appendix E on the CD. Additional case studies are available in journals and publications. In particular, *Interfaces* (published by INFORMS) is a rich source of diverse OR applications.

REFERENCES

Altier, W. J., *The Thinking Manager's Toolbox: Effective Processes for Problem Solving and Decision Making*, Oxford University Press, New York, 1999.

Checkland, P., *Systems Thinking, System Practice*, Wiley, New York, 1999.

Evans, J., *Creative Thinking in the Decision and Management Sciences*, South-Western Publishing, Cincinnati, 1991.

Gass, S., "Model World: Danger, Beware the User as a Modeler," *Interfaces*, Vol. 20, No. 3, pp. 60–64, 1990.

Morris, W., "On the Art of Modeling," *Management Science*, Vol. 13, pp. B707–B717, 1967.

Paulos, J.A., *Innumeracy: Mathematical Illiteracy and its Consequences*, Hill and Wang, New York, 1988.

Singh, Simon, *Fermat's Enigma*, Walker, New York, 1997.

Willemain, T. R., "Insights on Modeling from a Dozen Experts," *Operations Research*, Vol. 42, No. 2, pp. 213–222, 1994.

C H A P T E R 2

Modeling with Linear Programming

Chapter Guide. This chapter concentrates on model formulation and computations in linear programming (LP). It starts with the modeling and graphical solution of a two-variable problem which, though highly simplified, provides a concrete understanding of the basic concepts of LP and lays the foundation for the development of the general *simplex* algorithm in Chapter 3. To illustrate the use of LP in the real world, applications are formulated and solved in the areas of urban planning, currency arbitrage, investment, production planning and inventory control, gasoline blending, manpower planning, and scheduling. On the computational side, two distinct types of software are used in this chapter. (1) TORA, a totally menu-driven and self-documenting tutorial program, is designed to help you understand the basics of LP through interactive feedback. (2) Spreadsheet-based Excel Solver and the AMPL modeling language are commercial packages designed for practical problems.

The material in Sections 2.1 and 2.2 is crucial for understanding later LP developments in the book. You will find TORA's interactive graphical module especially helpful in conjunction with Section 2.2. Section 2.3 presents diverse LP applications, each followed by targeted problems.

Section 2.4 introduces the commercial packages Excel Solver and AMPL. Models in Section 2.3 are solved with AMPL and Solver, and all the codes are included in folder ch2Files. Additional Solver and AMPL models are included opportunely in the succeeding chapters, and a detailed presentation of AMPL syntax is given in Appendix A. A good way to learn AMPL and Solver is to experiment with the numerous models presented throughout the book and to try to adapt them to the end-of-section problems. The AMPL codes are cross-referenced with the material in Appendix A to facilitate the learning process.

The TORA, Solver, and AMPL materials have been deliberately compartmentalized either in separate sections or under the subheadings *TORA/Solver/AMPL moment* to minimize disruptions in the main text. Nevertheless, you are encouraged to work end-of-section problems on the computer. The reason is that, at times, a model may look "correct" until you try to obtain a solution, and only then will you discover that the formulation needs modifications.

This chapter includes summaries of 2 real-life applications, 12 solved examples, 2 Solver models, 4 AMPL models, 94 end-of-section problems, and 4 cases. The cases are in Appendix E on the CD. The AMPL/Excel/Solver/TORA programs are in folder ch2Files.

Real-Life Application–Frontier Airlines Purchases Fuel Economically

The fueling of an aircraft can take place at any of the stopovers along the flight route. Fuel price varies among the stopovers, and potential savings can be realized by loading extra fuel (called *tankering*) at a cheaper location for use on subsequent flight legs. The disadvantage of tankering is the excess burn of gasoline resulting from the extra weight. LP (and heuristics) is used to determine the optimum amount of tankering that balances the cost of excess burn against the savings in fuel cost. The study, carried out in 1981, resulted in net savings of about $350,000 per year. Case 1 in Chapter 24 on the CD provides the details of the study. Interestingly, with the recent rise in the cost of fuel, many airlines are now using LP-based tankering software to purchase fuel.

2.1 TWO-VARIABLE LP MODEL

This section deals with the graphical solution of a two-variable LP. Though two-variable problems hardly exist in practice, the treatment provides concrete foundations for the development of the general simplex algorithm presented in Chapter 3.

Example 2.1-1 (The Reddy Mikks Company)

Reddy Mikks produces both interior and exterior paints from two raw materials, $M1$ and $M2$. The following table provides the basic data of the problem:

	Tons of raw material per ton of		Maximum daily availability (tons)
	Exterior paint	*Interior paint*	
Raw material, $M1$	6	4	24
Raw material, $M2$	1	2	6
Profit per ton ($1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons.

Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

The LP model, as in any OR model, has three basic components.

1. Decision **variables** that we seek to determine.
2. **Objective** (goal) that we need to optimize (maximize or minimize).
3. **Constraints** that the solution must satisfy.

The proper definition of the decision variables is an essential first step in the development of the model. Once done, the task of constructing the objective function and the constraints becomes more straightforward.

For the Reddy Mikks problem, we need to determine the daily amounts to be produced of exterior and interior paints. Thus the variables of the model are defined as

$$x_1 = \text{Tons produced daily of exterior paint}$$

$$x_2 = \text{Tons produced daily of interior paint}$$

To construct the objective function, note that the company wants to *maximize* (i.e., increase as much as possible) the total daily profit of both paints. Given that the profits per ton of exterior and interior paints are 5 and 4 (thousand) dollars, respectively, it follows that

$$\text{Total profit from exterior paint} = 5x_1 \text{ (thousand) dollars}$$

$$\text{Total profit from interior paint} = 4x_2 \text{ (thousand) dollars}$$

Letting z represent the total daily profit (in thousands of dollars), the objective of the company is

$$\text{Maximize } z = 5x_1 + 4x_2$$

Next, we construct the constraints that restrict raw material usage and product demand. The raw material restrictions are expressed verbally as

$$\begin{pmatrix}\text{Usage of a raw material}\\ \text{by both paints}\end{pmatrix} \leq \begin{pmatrix}\text{Maximum raw material}\\ \text{availability}\end{pmatrix}$$

The daily usage of raw material $M1$ is 6 tons per ton of exterior paint and 4 tons per ton of interior paint. Thus

$$\text{Usage of raw material } M1 \text{ by exterior paint} = 6x_1 \text{ tons/day}$$

$$\text{Usage of raw material } M1 \text{ by interior paint} = 4x_2 \text{ tons/day}$$

Hence

$$\text{Usage of raw material } M1 \text{ by both paints} = 6x_1 + 4x_2 \text{ tons/day}$$

In a similar manner,

$$\text{Usage of raw material } M2 \text{ by both paints} = 1x_1 + 2x_2 \text{ tons/day}$$

Because the daily availabilities of raw materials $M1$ and $M2$ are limited to 24 and 6 tons, respectively, the associated restrictions are given as

$$6x_1 + 4x_2 \leq 24 \quad \text{(Raw material } M1\text{)}$$

$$x_1 + 2x_2 \leq 6 \quad \text{(Raw material } M2\text{)}$$

The first demand restriction stipulates that the excess of the daily production of interior over exterior paint, $x_2 - x_1$, should not exceed 1 ton, which translates to

$$x_2 - x_1 \leq 1 \qquad \text{(Market limit)}$$

The second demand restriction stipulates that the maximum daily demand of interior paint is limited to 2 tons, which translates to

$$x_2 \le 2 \text{ (Demand limit)}$$

An implicit (or "understood-to-be") restriction is that variables x_1 and x_2 cannot assume negative values. The **nonnegativity restrictions,** $x_1 \ge 0$, $x_2 \ge 0$, account for this requirement.

The complete Reddy Mikks model is

$$\text{Maximize } z = 5x_1 + 4x_2$$

subject to

$$6x_1 + 4x_2 \le 24 \quad (1)$$
$$x_1 + 2x_2 \le 6 \quad (2)$$
$$-x_1 + x_2 \le 1 \quad (3)$$
$$x_2 \le 2 \quad (4)$$
$$x_1, x_2 \ge 0 \quad (5)$$

Any values of x_1 and x_2 that satisfy *all* five constraints constitute a **feasible solution**. Otherwise, the solution is **infeasible**. For example, the solution, $x_1 = 3$ tons per day and $x_2 = 1$ ton per day, is feasible because it does not violate *any* of the constraints, including the nonnegativity restrictions. To verify this result, substitute $(x_1 = 3, x_2 = 1)$ in the left-hand side of each constraint. In constraint (1) we have $6x_1 + 4x_2 = 6 \times 3 + 4 \times 1 = 22$, which is less than the right-hand side of the constraint (= 24). Constraints 2 through 5 will yield similar conclusions (verify!). On the other hand, the solution $x_1 = 4$ and $x_2 = 1$ is infeasible because it does not satisfy constraint (1)—namely, $6 \times 4 + 4 \times 1 = 28$, which is larger than the right-hand side (= 24).

The goal of the problem is to find the best *feasible* solution, or the **optimum**, that maximizes the total profit. Before we can do that, we need to know how many *feasible* solutions the Reddy Mikks problem has. The answer, as we will see from the graphical solution in Section 2.2, is "an infinite number," which makes it impossible to solve the problem by enumeration. Instead, we need a systematic procedure that will locate the optimum solution in a finite number of steps. The graphical method in Section 2.2 and its algebraic generalization in Chapter 3 will explain how this can be accomplished.

Properties of the LP Model. In Example 2.1-1, the objective and the constraints are all linear functions. **Linearity** implies that the LP must satisfy three basic properties:

1. **Proportionality:** This property requires the contribution of each decision variable in both the objective function and the constraints to be *directly proportional* to the value of the variable. For example, in the Reddy Mikks model, the quantities $5x_1$ and $4x_2$ give the profits for producing x_1 and x_2 tons of exterior and in- 2.2
terior paint, respectively, with the unit profits per ton, 5 and 4, providing the constants of proportionality. If, on the other hand, Reddy Mikks grants some sort of quantity discounts when sales exceed certain amounts, then the profit will no longer be proportional to the production amounts, x_1 and x_2, and the profit function becomes nonlinear.

2. **Additivity:** This property requires the total contribution of all the variables in the objective function and in the constraints to be the direct sum of the individual contributions of each variable. In the Reddy Mikks model, the total profit equals the

sum of the two individual profit components. If, however, the two products *compete* for market share in such a way that an increase in sales of one adversely affects the other, then the additivity property is not satisfied and the model is no longer linear.

3. Certainty: All the objective and constraint coefficients of the LP model are deterministic. This means that they are known constants—a rare occurrence in real life, where data are more likely to be represented by probabilistic distributions. In essence, LP coefficients are average-value approximations of the probabilistic distributions. If the standard deviations of these distributions are sufficiently small, then the approximation is acceptable. Large standard deviations can be accounted for directly by using stochastic LP algorithms (Section 19.2.3) or indirectly by applying sensitivity analysis to the optimum solution (Section 3.6).

PROBLEM SET 2.1A

1. For the Reddy Mikks model, construct each of the following constraints and express it with a linear left-hand side and a constant right-hand side:
   - *(a) The daily demand for interior paint exceeds that of exterior paint by *at least* 1 ton.
   - (b) The daily usage of raw material $M2$ in tons is *at most* 6 and *at least* 3.
   - *(c) The demand for interior paint cannot be less than the demand for exterior paint.
   - (d) The minimum quantity that should be produced of both the interior and the exterior paint is 3 tons.
   - *(e) The proportion of interior paint to the total production of both interior and exterior paints must not exceed .5.
2. Determine the best *feasible* solution among the following (feasible and infeasible) solutions of the Reddy Mikks model:
   - (a) $x_1 = 1, x_2 = 4$.
   - (b) $x_1 = 2, x_2 = 2$.
   - (c) $x_1 = 3, x_2 = 1.5$.
   - (d) $x_1 = 2, x_2 = 1$.
   - (e) $x_1 = 2, x_2 = -1$.

*3. For the feasible solution $x_1 = 2, x_2 = 2$ of the Reddy Mikks model, determine the unused amounts of raw materials $M1$ and $M2$.

4. Suppose that Reddy Mikks sells its exterior paint to a single wholesaler at a quantity discount. The profit per ton is $5000 if the contractor buys no more than 2 tons daily and $4500 otherwise. Express the objective function mathematically. Is the resulting function linear?

2.2 GRAPHICAL LP SOLUTION

The graphical procedure includes two steps:

1. Determination of the feasible solution space.
2. Determination of the optimum solution from among all the feasible points in the solution space.

The procedure uses two examples to show how maximization and minimization objective functions are handled.

2.2.1 Solution of a Maximization Model

Example 2.2-1

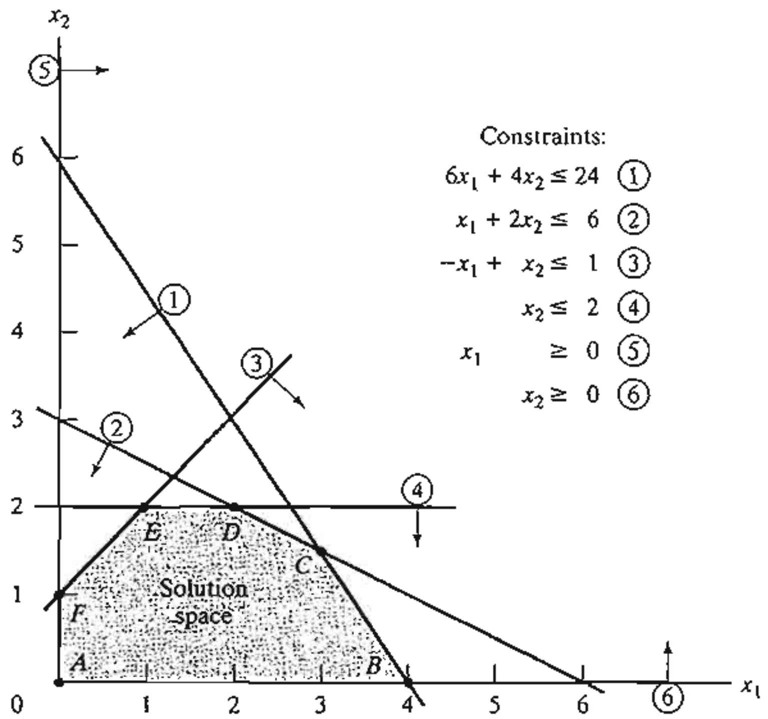
This example solves the Reddy Mikks model of Example 2.1-1.

Step 1. *Determination of the Feasible Solution Space:*
First, we account for the nonnegativity constraints $x_1 \geq 0$ and $x_2 \geq 0$. In Figure 2.1, the horizontal axis x_1 and the vertical axis x_2 represent the exterior- and interior-paint variables, respectively. Thus, the nonnegativity of the variables restricts the solution-space area to the first quadrant that lies above the x_1-axis and to the right of the x_2-axis.

To account for the remaining four constraints, first replace each inequality with an equation and then graph the resulting straight line by locating two distinct points on it. For example, after replacing $6x_1 + 4x_2 \leq 24$ with the straight line $6x_1 + 4x_2 = 24$, we can determine two distinct points by first setting $x_1 = 0$ to obtain $x_2 = \frac{24}{4} = 6$ and then setting $x_2 = 0$ to obtain $x_1 = \frac{24}{6} = 4$. Thus, the line passes through the two points (0, 6) and (4, 0), as shown by line (1) in Figure 2.1.

Next, consider the effect of the inequality. All it does is divide the (x_1, x_2)-plane into two half-spaces, one on each side of the graphed line. Only one of these two halves satisfies the inequality. To determine the correct side, choose (0, 0) as a *reference point*. If it satisfies the inequality, then the side in which it lies is the

FIGURE 2.1
Feasible space of the Reddy Mikks model



feasible half-space, otherwise the other side is. The use of the reference point $(0, 0)$ is illustrated with the constraint $6x_1 + 4x_2 \leq 24$. Because $6 \times 0 + 4 \times 0 = 0$ is less than 24, the half-space representing the inequality includes the origin (as shown by the arrow in Figure 2.1).

It is convenient computationally to select $(0, 0)$ as the reference point, unless the line happens to pass through the origin, in which case any other point can be used. For example, if we use the reference point $(6, 0)$, the left-hand side of the first constraint is $6 \times 6 + 4 \times 0 = 36$, which is larger than its right-hand side $(= 24)$, which means that the side in which $(6, 0)$ lies is not feasible for the inequality $6x_1 + 4x_2 \leq 24$. The conclusion is consistent with the one based on the reference point $(0, 0)$.

Application of the reference-point procedure to all the constraints of the model produces the constraints shown in Figure 2.1 (verify!). The **feasible solution space** of the problem represents the area in the first quadrant in which all the constraints are satisfied simultaneously. In Figure 2.1, any point in or on the boundary of the area *ABCDEF* is part of the feasible solution space. All points outside this area are infeasible.

TORA Moment.

The menu-driven TORA graphical LP module should prove helpful in reinforcing your understanding of how the LP constraints are graphed. Select Linear Programming from the MAIN menu. After inputting the model, select Solve ⇒ Graphical from the SOLVE/MODIFY menu. In the output screen, you will be able to experiment interactively with graphing the constraints one at a time, so you can see how each constraint affects the solution space.

Step 2. *Determination of the Optimum Solution:*

The feasible space in Figure 2.1 is delineated by the line segments joining the points *A, B, C, D, E*, and *F*. Any point within or on the boundary of the space *ABCDEF* is feasible. Because the feasible space *ABCDEF* consists of an *infinite* number of points, we need a systematic procedure to identify the optimum solution.

The determination of the optimum solution requires identifying the direction in which the profit function $z = 5x_1 + 4x_2$ increases (recall that we are *maximizing* z). We can do so by assigning *arbitrary* increasing values to z. For example, using $z = 10$ and $z = 15$ would be equivalent to graphing the two lines $5x_1 + 4x_2 = 10$ and $5x_1 + 4x_2 = 15$. Thus, the direction of increase in z is as shown Figure 2.2. The optimum solution occurs at C, which is the point in the solution space beyond which any further increase will put z outside the boundaries of *ABCDEF*.

The values of x_1 and x_2 associated with the optimum point C are determined by solving the equations associated with lines (1) and (2)—that is,

$$6x_1 + 4x_2 = 24$$
$$x_1 + 2x_2 = 6$$

The solution is $x_1 = 3$ and $x_2 = 1.5$ with $z = 5 \times 3 + 4 \times 1.5 = 21$. This calls for a daily product mix of 3 tons of exterior paint and 1.5 tons of interior paint. The associated daily profit is \$21,000.

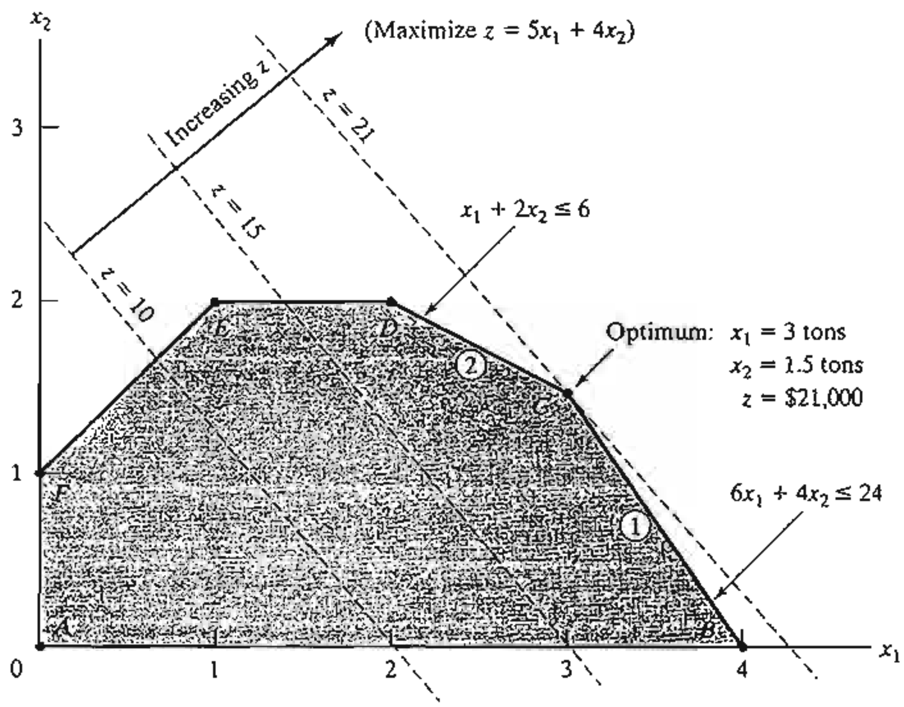


FIGURE 2.2
Optimum solution of the Reddy Mikks model

An important characteristic of the optimum LP solution is that it is *always* associated with a **corner point** of the solution space (where two lines intersect). This is true even if the objective function happens to be parallel to a constraint. For example, if the objective function is $z = 6x_1 + 4x_2$, which is parallel to constraint 1, we can always say that the optimum occurs at either corner point B or corner point C. Actually any point on the line segment BC will be an *alternative* optimum (see also Example 3.5-2), but the important observation here is that the line segment BC is totally defined by the *corner points* B and C.

TORA Moment.

You can use TORA interactively to see that the optimum is always associated with a corner point. From the output screen, you can click View/Modify Input Data to modify the objective coefficients and re-solve the problem graphically. You may use the following objective functions to test the proposed idea:

(a) $z = 5x_1 + x_2$

(b) $z = 5x_1 + 4x_2$

(c) $z = x_1 + 3x_2$

(d) $z = -x_1 + 2x_2$

(e) $z = -2x_1 + x_2$

(f) $z = -x_1 - x_2$

The observation that the LP optimum is always associated with a corner point means that the optimum solution can be found simply by enumerating all the corner points as the following table shows:

Corner point	(x_1, x_2)	z
A	(0, 0)	0
B	(4, 0)	20
C	**(3, 1.5)**	**21 (OPTIMUM)**
D	(2, 2)	18
E	(1, 2)	13
F	(0, 1)	4

As the number of constraints and variables increases, the number of corner points also increases, and the proposed enumeration procedure becomes less tractable computationally. Nevertheless, the idea shows that, from the standpoint of determining the LP optimum, the solution space $ABCDEF$ with its *infinite* number of solutions can, in fact, be replaced with a *finite* number of promising solution points—namely, the corner points, A, B, C, D, E, and F. This result is key for the development of the general algebraic algorithm, called the *simplex method*, which we will study in Chapter 3.

PROBLEM SET 2.2A

1. Determine the feasible space for each of the following independent constraints, given that $x_1, x_2 \geq 0$.
   *(a) $-3x_1 + x_2 \leq 6$.
   (b) $x_1 - 2x_2 \geq 5$.
   (c) $2x_1 - 3x_2 \leq 12$.
   *(d) $x_1 - x_2 \leq 0$.
   (e) $-x_1 + x_2 \geq 0$.
2. Identify the direction of increase in z in each of the following cases:
   *(a) Maximize $z = x_1 - x_2$.
   (b) Maximize $z = -5x_1 - 6x_2$.
   (c) Maximize $z = -x_1 + 2x_2$.
   *(d) Maximize $z = -3x_1 + x_2$.
3. Determine the solution space and the optimum solution of the Reddy Mikks model for each of the following independent changes:
   (a) The maximum daily demand for exterior paint is at most 2.5 tons.
   (b) The daily demand for interior paint is at least 2 tons.
   (c) The daily demand for interior paint is exactly 1 ton higher than that for exterior paint.
   (d) The daily availability of raw material $M1$ is at least 24 tons.
   (e) The daily availability of raw material $M1$ is at least 24 tons, and the daily demand for interior paint exceeds that for exterior paint by at least 1 ton.

4. A company that operates 10 hours a day manufactures two products on three sequential processes. The following table summarizes the data of the problem:

	Minutes per unit			
Product	*Process 1*	*Process 2*	*Process 3*	Unit profit
1	10	6	8	\$2
2	5	20	10	\$3

Determine the optimal mix of the two products.

*5. A company produces two products, A and B. The sales volume for A is at least 80% of the total sales of both A and B. However, the company cannot sell more than 100 units of A per day. Both products use one raw material, of which the maximum daily availability is 240 lb. The usage rates of the raw material are 2 lb per unit of A and 4 lb per unit of B. The profit units for A and B are \$20 and \$50, respectively. Determine the optimal product mix for the company.

6. Alumco manufactures aluminum sheets and aluminum bars. The maximum production capacity is estimated at either 800 sheets or 600 bars per day. The maximum daily demand is 550 sheets and 580 bars. The profit per ton is \$40 per sheet and \$35 per bar. Determine the optimal daily production mix.

*7. An individual wishes to invest \$5000 over the next year in two types of investment: Investment A yields 5% and investment B yields 8%. Market research recommends an allocation of at least 25% in A and at most 50% in B. Moreover, investment in A should be at least half the investment in B. How should the fund be allocated to the two investments?

8. The Continuing Education Division at the Ozark Community College offers a total of 30 courses each semester. The courses offered are usually of two types: practical, such as woodworking, word processing, and car maintenance; and humanistic, such as history, music, and fine arts. To satisfy the demands of the community, at least 10 courses of each type must be offered each semester. The division estimates that the revenues of offering practical and humanistic courses are approximately \$1500 and \$1000 per course, respectively.
   (a) Devise an optimal course offering for the college.
   (b) Show that the worth per additional course is \$1500, which is the same as the revenue per practical course. What does this result mean in terms of offering additional courses?

9. ChemLabs uses raw materials I and II to produce two domestic cleaning solutions, A and B. The daily availabilities of raw materials I and II are 150 and 145 units, respectively. One unit of solution A consumes .5 unit of raw material I and .6 unit of raw material II, and one unit of solution B uses .5 unit of raw material I and .4 unit of raw material II. The profits per unit of solutions A and B are \$8 and \$10, respectively. The daily demand for solution A lies between 30 and 150 units, and that for solution B between 40 and 200 units. Find the optimal production amounts of A and B.

10. In the Ma-and-Pa grocery store, shelf space is limited and must be used effectively to increase profit. Two cereal items, Grano and Wheatie, compete for a total shelf space of 60 ft^2. A box of Grano occupies .2 ft^2 and a box of Wheatie needs .4 ft^2. The maximum daily demands of Grano and Wheatie are 200 and 120 boxes, respectively. A box of Grano nets \$1.00 in profit and a box of Wheatie \$1.35. Ma-and-Pa thinks that because the unit profit of Wheatie is 35% higher than that of Grano, Wheatie should be allocated

35% more space than Grano, which amounts to allocating about 57% to Wheatie and 43% to Grano. What do you think?

11. Jack is an aspiring freshman at Ulern University. He realizes that "all work and no play make Jack a dull boy." As a result, Jack wants to apportion his available time of about 10 hours a day between work and play. He estimates that play is twice as much fun as work. He also wants to study at least as much as he plays. However, Jack realizes that if he is going to get all his homework assignments done, he cannot play more than 4 hours a day. How should Jack allocate his time to maximize his pleasure from both work and play?

12. Wild West produces two types of cowboy hats. A type 1 hat requires twice as much labor time as a type 2. If the all available labor time is dedicated to Type 2 alone, the company can produce a total of 400 Type 2 hats a day. The respective market limits for the two types are 150 and 200 hats per day. The profit is $8 per Type 1 hat and $5 per Type 2 hat. Determine the number of hats of each type that would maximize profit.

13. Show & Sell can advertise its products on local radio and television (TV). The advertising budget is limited to $10,000 a month. Each minute of radio advertising costs $15 and each minute of TV commercials $300. Show & Sell likes to advertise on radio at least twice as much as on TV. In the meantime, it is not practical to use more than 400 minutes of radio advertising a month. From past experience, advertising on TV is estimated to be 25 times as effective as on radio. Determine the optimum allocation of the budget to radio and TV advertising.

***14.** Wyoming Electric Coop owns a steam-turbine power-generating plant. Because Wyoming is rich in coal deposits, the plant generates its steam from coal. This, however, may result in emission that does not meet the Environmental Protection Agency standards. EPA regulations limit sulfur dioxide discharge to 2000 parts per million per ton of coal burned and smoke discharge from the plant stacks to 20 lb per hour. The Coop receives two grades of pulverized coal, $C1$ and $C2$, for use in the steam plant. The two grades are usually mixed together before burning. For simplicity, it can be assumed that the amount of sulfur pollutant discharged (in parts per million) is a weighted average of the proportion of each grade used in the mixture. The following data are based on consumption of 1 ton per hour of each of the two coal grades.

Coal grade	Sulfur discharge in parts per million	Smoke discharge in lb per hour	Steam generated in lb per hour
$C1$	1800	2.1	12,000
$C2$	2100	.9	9,000

(a) Determine the optimal ratio for mixing the two coal grades.

(b) Determine the effect of relaxing the smoke discharge limit by 1 lb on the amount of generated steam per hour.

15. Top Toys is planning a new radio and TV advertising campaign. A radio commercial costs $300 and a TV ad costs $2000. A total budget of $20,000 is allocated to the campaign. However, to ensure that each medium will have at least one radio commercial and one TV ad, the most that can be allocated to either medium cannot exceed 80% of the total budget. It is estimated that the first radio commercial will reach 5000 people, with each additional commercial reaching only 2000 new ones. For TV, the first ad will reach 4500 people and each additional ad an additional 3000. How should the budgeted amount be allocated between radio and TV?

16. The Burroughs Garment Company manufactures men's shirts and women's blouses for Walmark Discount Stores. Walmark will accept all the production supplied by Burroughs. The production process includes cutting, sewing, and packaging. Burroughs employs 25 workers in the cutting department, 35 in the sewing department, and 5 in the packaging department. The factory works one 8-hour shift, 5 days a week. The following table gives the time requirements and profits per unit for the two garments:

	Minutes per unit			
Garment	*Cutting*	*Sewing*	*Packaging*	Unit profit ($)
Shirts	20	70	12	8
Blouses	60	60	4	12

Determine the optimal weekly production schedule for Burroughs.

17. A furniture company manufactures desks and chairs. The sawing department cuts the lumber for both products, which is then sent to separate assembly departments. Assembled items are sent for finishing to the painting department. The daily capacity of the sawing department is 200 chairs or 80 desks. The chair assembly department can produce 120 chairs daily and the desk assembly department 60 desks daily. The paint department has a daily capacity of either 150 chairs or 110 desks. Given that the profit per chair is $50 and that of a desk is $100, determine the optimal production mix for the company.

***18.** An assembly line consisting of three consecutive stations produces two radio models: HiFi-1 and HiFi-2. The following table provides the assembly times for the three workstations.

	Minutes per unit	
Workstation	*HiFi-1*	*HiFi-2*
1	6	4
2	5	5
3	4	6

The daily maintenance for stations 1, 2, and 3 consumes 10%, 14%, and 12%, respectively, of the maximum 480 minutes available for each station each day. Determine the optimal product mix that will minimize the idle (or unused) times in the three workstations.

19. *TORA Experiment.* Enter the following LP into TORA and select the graphic solution mode to reveal the LP graphic screen.

$$\text{Minimize } z = 3x_1 + 8x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\geq 8 \\ 2x_1 - 3x_2 &\leq 0 \\ x_1 + 2x_2 &\leq 30 \\ 3x_1 - x_2 &\geq 0 \\ x_1 &\leq 10 \\ x_2 &\geq 9 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Next, on a sheet of paper, graph and scale the x_1- and x_2-axes for the problem (you may also click Print Graph on the top of the right window to obtain a ready-to-use scaled sheet). Now, graph a constraint manually on the prepared sheet, then click it on the left window of the screen to check your answer. Repeat the same for each constraint and then terminate the procedure with a graph of the objective function. The suggested process is designed to test and reinforce your understanding of the graphical LP solution through immediate feedback from TORA.

20. *TORA Experiment.* Consider the following LP model:

$$\text{Maximize } z = 5x_1 + 4x_2$$

subject to

$$\begin{aligned} 6x_1 + 4x_2 &\leq 24 \\ 6x_1 + 3x_2 &\leq 22.5 \\ x_1 + x_2 &\leq 5 \\ x_1 + 2x_2 &\leq 6 \\ -x_1 + x_2 &\leq 1 \\ x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

In LP, a constraint is said to be *redundant* if its removal from the model leaves the feasible solution space unchanged. Use the graphical facility of TORA to identify the redundant constraints, then show that their removal (simply by not graphing them) does not affect the solution space or the optimal solution.

21. *TORA Experiment.* In the Reddy Mikks model, use TORA to show that the removal of the raw material constraints (constraints 1 and 2) would result in an *unbounded solution space.* What can be said in this case about the optimal solution of the model?

22. *TORA Experiment.* In the Reddy Mikks model, suppose that the following constraint is added to the problem.

$$x_2 \geq 3$$

Use TORA to show that the resulting model has conflicting constraints that cannot be satisfied simultaneously and hence it has *no feasible solution.*

2.2.2 Solution of a Minimization Model

Example 2.2-2 (Diet Problem)

Ozark Farms uses at least 800 lb of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions:

	lb per lb of feedstuff		
Feedstuff	*Protein*	*Fiber*	Cost ($/lb)
Corn	.09	.02	.30
Soybean meal	.60	.06	.90

The dietary requirements of the special feed are at least 30% protein and at most 5% fiber. Ozark Farms wishes to determine the daily minimum-cost feed mix.

Because the feed mix consists of corn and soybean meal, the decision variables of the model are defined as

$$x_1 = \text{lb of corn in the daily mix}$$
$$x_2 = \text{lb of soybean meal in the daily mix}$$

The objective function seeks to minimize the total daily cost (in dollars) of the feed mix and is thus expressed as

$$\text{Minimize } z = .3x_1 + .9x_2$$

The constraints of the model reflect the daily amount needed and the dietary requirements. Because Ozark Farms needs at least 800 lb of feed a day, the associated constraint can be expressed as

$$x_1 + x_2 \geq 800$$

As for the protein dietary requirement constraint, the amount of protein included in x_1 lb of corn and x_2 lb of soybean meal is $(.09x_1 + .6x_2)$ lb. This quantity should equal at least 30% of the total feed mix $(x_1 + x_2)$ lb—that is,

$$.09x_1 + .6x_2 \geq .3(x_1 + x_2)$$

In a similar manner, the fiber requirement of at most 5% is constructed as

$$.02x_1 + .06x_2 \leq .05(x_1 + x_2)$$

The constraints are simplified by moving the terms in x_1 and x_2 to the left-hand side of each inequality, leaving only a constant on the right-hand side. The complete model thus becomes

$$\text{minimize } z = .3x_1 + .9x_2$$

subject to

$$\begin{aligned} x_1 + \quad x_2 &\geq 800 \\ .21x_1 - .30x_2 &\leq 0 \\ .03x_1 - .01x_2 &\geq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Figure 2.3 provides the graphical solution of the model. Unlike those of the Reddy Mikks model (Example 2.2-1), the second and third constraints pass through the origin. To plot the associated straight lines, we need one additional point, which can be obtained by assigning a value to one of the variables and then solving for the other variable. For example, in the second constraint, $x_1 = 200$ will yield $.21 \times 200 - .3x_2 = 0$, or $x_2 = 140$. This means that the straight line $.21x_1 - .3x_2 = 0$ passes through $(0, 0)$ and $(200, 140)$. Note also that $(0, 0)$ cannot be used as a reference point for constraints 2 and 3, because both lines pass through the origin. Instead, any other point [e.g., $(100, 0)$ or $(0, 100)$] can be used for that purpose.

Solution:

Because the present model seeks the minimization of the objective function, we need to reduce the value of z as much as possible in the direction shown in Figure 2.3. The optimum solution is the intersection of the two lines $x_1 + x_2 = 800$ and $.21x_1 - .3x_2 = 0$, which yields $x_1 = 470.59$ lb and $x_2 = 329.41$ lb. The associated minimum cost of the feed mix is $z = .3 \times 470.59 + .9 \times 329.42 =$ \$437.65 per day.

Remarks. We need to take note of the way the constraints of the problem are constructed. Because the model is minimizing the total cost, one may argue that the solution will seek exactly 800 tons of feed. Indeed, this is what the optimum solution given above does. Does this mean then that the first constraint can be deleted altogether simply by including the amount 800 tons

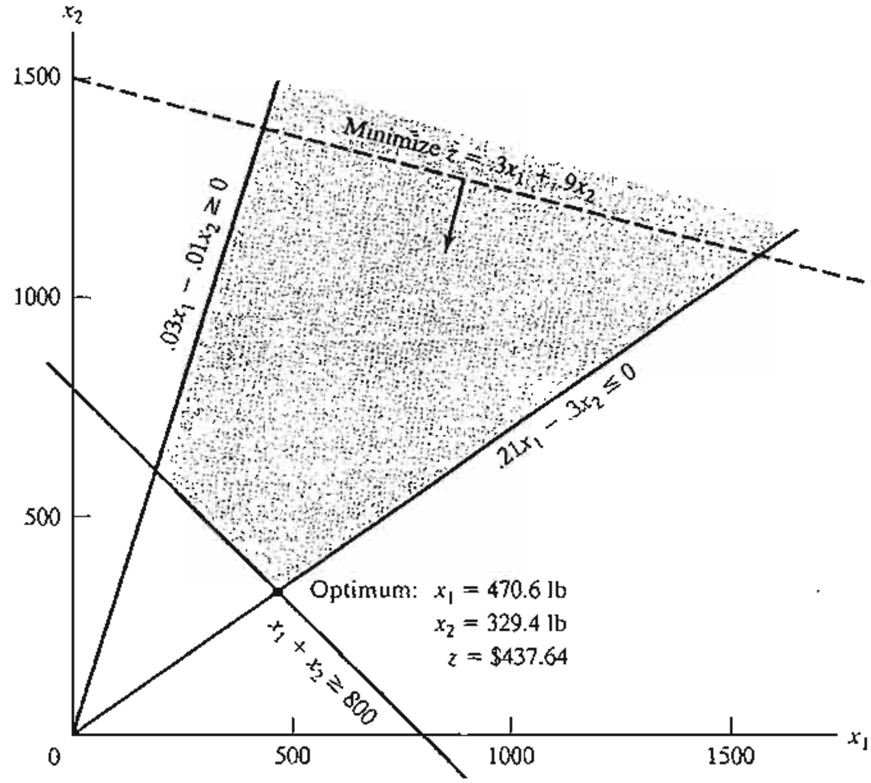


FIGURE 2.3
Graphical solution of the diet model

in the remaining constraints? To find the answer, we state the new protein and fiber constraints as

$$.09x_1 + .6x_2 \geq .3 \times 800$$
$$.02x_1 + .06x_2 \leq .05 \times 800$$

or

$$.09x_1 + .6x_2 \geq 240$$
$$.02x_1 + .06x_2 \leq 40$$

The new formulation yields the solution $x_1 = 0$, and $x_2 = 400$ lb (verify with TORA!), which does not satisfy the *implied* requirement for 800 lb of feed. This means that the constraint $x_1 + x_2 \geq 800$ must be used explicitly and that the protein and fiber constraints must remain exactly as given originally.

Along the same line of reasoning, one may be tempted to replace $x_1 + x_2 \geq 800$ with $x_1 + x_2 = 800$. In the present example, the two constraints yield the same answer. But in general this may not be the case. For example, suppose that the daily mix must include at least 500 lb of corn. In this case, the optimum solution will call for using 500 lb of corn and 350 lb of soybean (verify with TORA!), which is equivalent to a daily feed mix of $500 + 350 = 850$ lb. Imposing the equality constraint a priori will lead to the conclusion that the problem has no feasible solution (verify with TORA!). On the other hand, the use of the inequality is inclusive of the equality case, and hence its use does not prevent the model from producing exactly 800 lb of feed mix, should the remaining constraints allow it. The conclusion is that we should not "pre-guess" the solution by imposing the additional equality restriction, and we should always use inequalities unless the situation explicitly stipulates the use of equalities.

PROBLEM SET 2.2B

1. Identify the direction of decrease in z in each of the following cases:
   *(a) Minimize $z = 4x_1 - 2x_2$.
   (b) Minimize $z = -3x_1 + x_2$.
   (c) Minimize $z = -x_1 - 2x_2$.
2. For the diet model, suppose that the daily availability of corn is limited to 450 lb. Identify the new solution space, and determine the new optimum solution. 2.
3. For the diet model, what type of optimum solution would the model yield if the feed mix should not exceed 800 lb a day? Does the solution make sense?
4. John must work at least 20 hours a week to supplement his income while attending school. He has the opportunity to work in two retail stores. In store 1, he can work between 5 and 12 hours a week, and in store 2 he is allowed between 6 and 10 hours. Both stores pay the same hourly wage. In deciding how many hours to work in each store, John wants to base his decision on work stress. Based on interviews with present employees, John estimates that, on an ascending scale of 1 to 10, the stress factors are 8 and 6 at stores 1 and 2, respectively. Because stress mounts by the hour, he assumes that the total stress for each store at the end of the week is proportional to the number of hours he works in the store. How many hours should John work in each store?

*5. OilCo is building a refinery to produce four products: diesel, gasoline, lubricants, and jet fuel. The minimum demand (in bbl/day) for each of these products is 14,000, 30,000, 10,000, and 8,000, respectively. Iran and Dubai are under contract to ship crude to OilCo. Because of the production quotas specified by OPEC (Organization of Petroleum Exporting Countries) the new refinery can receive at least 40% of its crude from Iran and the remaining amount from Dubai. OilCo predicts that the demand and crude oil quotas will remain steady over the next ten years. 2.

   The specifications of the two crude oils lead to different product mixes: One barrel of Iran crude yields .2 bbl of diesel, .25 bbl of gasoline, .1 bbl of lubricant, and .15 bbl of jet fuel. The corresponding yields from Dubai crude are .1, .6, .15, and .1, respectively. OilCo needs to determine the minimum capacity of the refinery (in bbl/ day).

6. Day Trader wants to invest a sum of money that would generate an annual yield of at least $10,000. Two stock groups are available: blue chips and high tech, with average annual yields of 10% and 25%, respectively. Though high-tech stocks provide higher yield, they are more risky, and Trader wants to limit the amount invested in these stocks to no more than 60% of the total investment. What is the minimum amount Trader should invest in each stock group to accomplish the investment goal?

*7. An industrial recycling center uses two scrap aluminum metals, A and B, to produce a special alloy. Scrap A contains 6% aluminum, 3% silicon, and 4% carbon. Scrap B has 3% aluminum, 6% silicon, and 3% carbon. The costs per ton for scraps A and B are $100 and $80, respectively. The specifications of the special alloy require that (1) the aluminum content must be at least 3% and at most 6%, (2) the silicon content must lie between 3%

and 5%, and (3) the carbon content must be between 3% and 7%. Determine the optimum mix of the scraps that should be used in producing 1000 tons of the alloy.

8. *TORA Experiment.* Consider the Diet Model and let the objective function be given as

$$\text{Minimize } z = .8x_1 + .8x_2$$

Use TORA to show that the optimum solution is associated with *two* distinct corner points and that both points yield the same objective value. In this case, the problem is said to have *alternative optima.* Explain the conditions leading to this situation and show that, in effect, the problem has an infinite number of alternative optima, then provide a formula for determining all such solutions.

2.3 SELECTED LP APPLICATIONS

This section presents realistic LP models in which the definition of the variables and the construction of the objective function and constraints are not as straightforward as in the case of the two-variable model. The areas covered by these applications include the following:

1. Urban planning.
2. Currency arbitrage.
3. Investment.
4. Production planning and inventory control.
5. Blending and oil refining.
6. Manpower planning.

Each model is fully developed and its optimum solution is analyzed and interpreted.

2.3.1 Urban Planning[1]

Urban planning deals with three general areas: (1) building new housing developments, (2) upgrading inner-city deteriorating housing and recreational areas, and (3) planning public facilities (such as schools and airports). The constraints associated with these projects are both economic (land, construction, financing) and social (schools, parks, income level). The objectives in urban planning vary. In new housing developments, profit is usually the motive for undertaking the project. In the remaining two categories, the goals involve social, political, economic, and cultural considerations. Indeed, in a publicized case in 2004, the mayor of a city in Ohio wanted to condemn an old area of the city to make way for a luxury housing development. The motive was to increase tax collection to help alleviate budget shortages. The example presented in this section is fashioned after the Ohio case.

[1]This section is based on Laidlaw (1972).

Example 2.3-1 (Urban Renewal Model)

The city of Erstville is faced with a severe budget shortage. Seeking a long-term solution, the city council votes to improve the tax base by condemning an inner-city housing area and replacing it with a modern development.

The project involves two phases: (1) demolishing substandard houses to provide land for the new development, and (2) building the new development. The following is a summary of the situation.

1. As many as 300 substandard houses can be demolished. Each house occupies a .25-acre lot. The cost of demolishing a condemned house is $2000.
2. Lot sizes for new single-, double-, triple-, and quadruple-family homes (units) are .18, .28, .4, and .5 acre, respectively. Streets, open space, and utility easements account for 15% of available acreage.
3. In the new development the triple and quadruple units account for at least 25% of the total. Single units must be at least 20% of all units and double units at least 10%.
4. The tax levied per unit for single, double, triple, and quadruple units is $1,000, $1,900, $2,700, and $3,400, respectively.
5. The construction cost per unit for single-, double-, triple-, and quadruple- family homes is $50,000, $70,000, $130,000, and $160,000, respectively. Financing through a local bank can amount to a maximum of $15 million.

How many units of each type should be constructed to maximize tax collection?

Mathematical Model: Besides determining the number of units to be constructed of each type of housing, we also need to decide how many houses must be demolished to make room for the new development. Thus, the variables of the problem can be defined as follows:

x_1 = Number of units of single-family homes

x_2 = Number of units of double-family homes

x_3 = Number of units of triple-family homes

x_4 = Number of units of quadruple-family homes

x_5 = Number of old homes to be demolished

The objective is to maximize total tax collection from all four types of homes—that is,

$$\text{Maximize } z = 1000x_1 + 1900x_2 + 2700x_3 + 3400x_4$$

The first constraint of the problem deals with land availability.

$$\begin{pmatrix}\text{Acreage used for new} \\ \text{home construction}\end{pmatrix} \leq \begin{pmatrix}\text{Net available} \\ \text{acreage}\end{pmatrix}$$

From the data of the problem we have

$$\text{Acreage needed for new homes} = .18x_1 + .28x_2 + .4x_3 + .5x_4$$

To determine the available acreage, each demolished home occupies a .25-acre lot, thus netting $.25x_5$ acres. Allowing for 15% open space, streets, and easements, the net acreage available is $.85(.25x_5) = .2125x_5$. The resulting constraint is

$$.18x_1 + .28x_2 + .4x_3 + .5x_4 \leq .2125x_5$$

or

$$.18x_1 + .28x_2 + .4x_3 + .5x_4 - .2125x_5 \leq 0$$

The number of demolished homes cannot exceed 300, which translates to

$$x_5 \leq 300$$

Next we add the constraints limiting the number of units of each home type.

$$(\text{Number of single units}) \geq (20\% \text{ of all units})$$

$$(\text{Number of double units}) \geq (10\% \text{ of all units})$$

$$(\text{Number of triple and quadruple units}) \geq (25\% \text{ of all units})$$

These constraints translate mathematically to

$$x_1 \geq .2(x_1 + x_2 + x_3 + x_4)$$

$$x_2 \geq .1(x_1 + x_2 + x_3 + x_4)$$

$$x_3 + x_4 \geq .25(x_1 + x_2 + x_3 + x_4)$$

The only remaining constraint deals with keeping the demolishition/construction cost within the allowable budget—that is,

$$(\text{Construction and demolition cost}) \leq (\text{Available budget})$$

Expressing all the costs in thousands of dollars, we get

$$(50x_1 + 70x_2 + 130x_3 + 160x_4) + 2x_5 \leq 15000$$

The complete model thus becomes

$$\text{Maximize } z = 1000x_1 + 1900x_2 + 2700x_3 + 3400x_4$$

subject to

$$.18x_1 + .28x_2 + .4x_3 + .5x_4 - .2125x_5 \leq 0$$

$$x_5 \leq 300$$

$$-.8x_1 + .2x_2 + .2x_3 + .2x_4 \leq 0$$

$$.1x_1 - .9x_2 + .1x_3 + .1x_4 \leq 0$$

$$.25x_1 + .25x_2 - .75x_3 - .75x_4 \leq 0$$

$$50x_1 + 70x_2 + 130x_3 + 160x_4 + 2x_5 \leq 15000$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Solution:

The optimum solution (using file amplEX2.3-1.txt or solverEx2.3-1.xls) is:

Total tax collection = z = \$343,965
Number of single homes = $x_1 = 35.83 \approx 36$ units
Number of double homes = $x_2 = 98.53 \approx 99$ units
Number of triple homes = $x_3 = 44.79 \approx 45$ units
Number of quadruple homes = $x_4 = 0$ units
Number of homes demolished = $x_5 = 244.49 \approx 245$ units

Remarks. Linear programming does not guarantee an integer solution automatically, and this is the reason for rounding the continuous values to the closest integer. The rounded solution calls for constructing 180 (= 36 + 99 + 45) units and demolishing 245 old homes, which yields \$345,600 in taxes. Keep in mind, however, that, in general, the rounded solution may not be feasible. In fact, the current rounded solution violates the budget constraint by \$70,000 (verify!). Interestingly, the true optimum integer solution (using the algorithms in Chapter 9) is $x_1 = 36, x_2 = 98, x_3 = 45, x_4 = 0$, and $x_5 = 245$ with z = \$343,700. Carefully note that the rounded solution yields a better objective value, which appears contradictory. The reason is that the rounded solution calls for producing an extra double home, which is feasible only if the budget is increased by \$70,000.

PROBLEM SET 2.3A

1. A realtor is developing a rental housing and retail area. The housing area consists of efficiency apartments, duplexes, and single-family homes. Maximum demand by potential renters is estimated to be 500 efficiency apartments, 300 duplexes, and 250 single-family homes, but the number of duplexes must equal at least 50% of the number of efficiency apartments and single homes. Retail space is proportionate to the number of home units at the rates of at least 10 ft^2, 15 ft^2, and 18 ft^2 for efficiency, duplex, and single family units, respectively. However, land availability limits retail space to no more than 10,000 ft^2. The monthly rental income is estimated at \$600, \$750, and \$1200 for efficiency-, duplex-, and single-family units, respectively. The retail space rents for \$100/ft^2. Determine the optimal retail space area and the number of family residences.
2. The city council of Fayetteville is in the process of approving the construction of a new 200,000-ft^2 convention center. Two sites have been proposed, and both require exercising the "eminent domain" law to acquire the property. The following table provides data about proposed (contiguous) properties in both sites together with the acquisition cost.

	Site 1		Site 2	
Property	*Area* (1000 ft^2)	*Cost* (1000 \$)	*Area* (1000 ft^2)	*Cost* (1000 \$)
1	20	1,000	80	2,800
2	50	2,100	60	1,900
3	50	2,350	50	2,800
4	30	1,850	70	2,500
5	60	2,950		

Partial acquisition of property is allowed. At least 75% of property 4 must be acquired if site 1 is selected, and at least 50% of property 3 must be acquired if site 2 is selected.

Although site 1 property is more expensive (on a per ft^2 basis), the construction cost is less than at site 2, because the infrastructure at site 1 is in a much better shape. Construction cost is \$25 million at site 1 and \$27 million at site 2. Which site should be selected, and what properties should be acquired?

*3. A city will undertake five urban renewal housing projects over the next five years. Each project has a different starting year and a different duration. The following table provides the basic data of the situation:

	Year 1	Year 2	Year 3	Year 4	Year 5	Cost (million \$)	Annual income (million \$)
Project 1	Start		End			5.0	.05
Project 2		Start			End	8.0	.07
Project 3	Start				End	15.0	.15
Project 4			Start	End		1.2	.02
Budget (million \$)	3.0	6.0	7.0	7.0	7.0		

Projects 1 and 4 must be finished completely within their durations. The remaining two projects can be finished partially within budget limitations, if necessary. However, each project must be at least 25% completed within its duration. At the end of each year, the completed section of a project is immediately occupied by tenants and a proportional amount of income is realized. For example, if 40% of project 1 is completed in year 1 and 60% in year 3, the associated income over the five-year planning horizon is $.4 \times \$50{,}000$ (for year 2) $+ .4 \times \$50{,}000$ (for year 3) $+ (.4 + .6) \times \$50{,}000$ (for year 4) $+ (.4 + .6) \times \$50{,}000$ (for year 5) $= (4 \times .4 + 2 \times .6) \times \$50{,}000$. Determine the optimal schedule for the projects that will maximize the total income over the five-year horizon. For simplicity, disregard the time value of money.

4. The city of Fayetteville is embarking on an urban renewal project that will include lower- and middle-income row housing, upper-income luxury apartments, and public housing. The project also includes a public elementary school and retail facilities. The size of the elementary school (number of classrooms) is proportional to the number of pupils, and the retail space is proportional to the number of housing units. The following table provides the pertinent data of the situation:

	Lower income	Middle income	Upper income	Public housing	School room	Retail unit
Minimum number of units	100	125	75	300		0
Maximum number of units	200	190	260	600		25
Lot size per unit (acre)	.05	.07	.03	.025	.045	.1
Average number of pupils per unit	1.3	1.2	.5	1.4		
Retail demand per unit (acre)	.023	.034	.046	.023	.034	
Annual income per unit(\$)	7000	12,000	20,000	5000	—	15,000

The new school can occupy a maximum space of 2 acres at the rate of at most 25 pupils per room. The operating annual cost per school room is \$10,000. The project will be located on a 50-acre vacant property owned by the city. Additionally, the project can make use of an adjacent property occupied by 200 condemned slum homes. Each condemned home occupies .25 acre. The cost of buying and demolishing a slum unit is \$7000. Open space, streets, and parking lots consume 15% of total available land.

Develop a linear program to determine the optimum plan for the project.

5. Realco owns 800 acres of undeveloped land on a scenic lake in the heart of the Ozark Mountains. In the past, little or no regulation was imposed upon new developments around the lake. The lake shores are now dotted with vacation homes, and septic tanks, most of them improperly installed, are in extensive use. Over the years, seepage from the septic tanks led to severe water pollution. To curb further degradation of the lake, county officials have approved stringent ordinances applicable to all future developments: (1) Only single-, double-, and triple-family homes can be constructed, with single-family homes accounting for at least 50% of the total. (2) To limit the number of septic tanks, minimum lot sizes of 2, 3, and 4 acres are required for single-, double-, and triple-family homes, respectively. (3) Recreation areas of 1 acre each must be established at the rate of one area per 200 families. (4) To preserve the ecology of the lake, underground water may not be pumped out for house or garden use. The president of Realco is studying the possibility of developing the 800-acre property. The new development will include single-, double-, and triple-family homes. It is estimated that 15% of the acreage will be allocated to streets and utility easements. Realco estimates the returns from the different housing units as follows:

Housing unit	Single	Double	Triple
Net return per unit ($)	10,000	12,000	15,000

The cost of connecting water service to the area is proportionate to the number of units constructed. However, the county charges a minimum of $100,000 for the project. Additionally, the expansion of the water system beyond its present capacity is limited to 200,000 gallons per day during peak periods. The following data summarize the water service connection cost as well as the water consumption, assuming an average size family:

Housing unit	Single	Double	Triple	Recreation
Water service connection cost per unit ($)	1000	1200	1400	800
Water consumption per unit (gal/day)	400	600	840	450

Develop an optimal plan for Realco.

6. Consider the Realco model of Problem 5. Suppose that an additional 100 acres of land can be purchased for $450,000, which will increase the total acreage to 900 acres. Is this a profitable deal for Realco?

2.3.2 Currency Arbitrage[2]

In today's global economy, a multinational company must deal with currencies of the countries in which it operates. Currency arbitrage, or simultaneous purchase and sale of currencies in different markets, offers opportunities for advantageous movement of money from one currency to another. For example, converting £1000 to U.S. dollars in 2001 with an exchange rate of $1.60 to £1 will yield $1600. Another way of making the conversion is to first change the British pound to Japanese yen and then convert the yen to U.S. dollars using the 2001 exchange rates of £1 = ¥175 and $1 = ¥105. The

[2]This section is based on J. Kornbluth and G. Salkin (1987, Chapter 6).

resulting dollar amount is $\frac{(£1{,}000 \times ¥175)}{¥105} = \$1{,}666.67$. This example demonstrates the advantage of converting the British money first to Japanese yen and then to dollars. This section shows how the arbitrage problem involving many currencies can be formulated and solved as a linear program.

Example 2.3-2 (Currency Arbitrage Model)

Suppose that a company has a total of 5 million dollars that can be exchanged for euros (€), British pounds (£), yen (¥), and Kuwaiti dinars (KD). Currency dealers set the following limits on the amount of any single transaction: 5 million dollars, 3 million euros, 3.5 million pounds, 100 million yen, and 2.8 million KDs. The table below provides typical spot exchange rates. The bottom diagonal rates are the reciprocal of the top diagonal rates. For example, rate(€ → \$) = 1/rate(\$ → €) = 1/.769 = 1.30.

	\$	€	£	¥	KD
\$	1	.769	.625	105	.342
€	$\frac{1}{.769}$	1	.813	137	.445
£	$\frac{1}{.625}$	$\frac{1}{.813}$	1	169	.543
¥	$\frac{1}{105}$	$\frac{1}{137}$	$\frac{1}{169}$	1	.0032
KD	$\frac{1}{.342}$	$\frac{1}{.445}$	$\frac{1}{.543}$	$\frac{1}{.0032}$	1

Is it possible to increase the dollar holdings (above the initial \$5 million) by circulating currencies through the currency market?

Mathematical Model: The situation starts with \$5 million. This amount goes through a number of conversions to other currencies before ultimately being reconverted to dollars. The problem thus seeks determining the amount of each conversion that will maximize the total dollar holdings.

For the purpose of developing the model and simplifying the notation, the following numeric code is used to represent the currencies.

Currency	\$	€	£	¥	KD
Code	1	2	3	4	5

Define

$$x_{ij} = \text{Amount in currency } i \text{ converted to currency } j,\ i \text{ and } j = 1, 2, \ldots, 5$$

For example, x_{12} is the dollar amount converted to euros and x_{51} is the KD amount converted to dollars. We further define two additional variables representing the input and the output of the arbitrage problem:

$$I = \text{Initial dollar amount } (= \$5 \text{ million})$$
$$y = \text{Final dollar holdings (to be determined from the solution)}$$

Our goal is to determine the maximum final dollar holdings, y, subject to the currency flow restrictions and the maximum limits allowed for the different transactions.

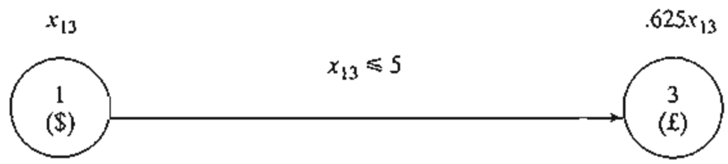


FIGURE 2.4
Definition of the input/output variable, x_{13}, between \$ and £

We start by developing the constraints of the model. Figure 2.4 demonstrates the idea of converting dollars to pounds. The dollar amount x_{13} at originating currency 1 is converted to $.625x_{13}$ pounds at end currency 3. At the same time, the transacted dollar amount cannot exceed the limit set by the dealer, $x_{13} \le 5$.

To conserve the flow of money from one currency to another, each currency must satisfy the following input-output equation:

$$\begin{pmatrix}\text{Total sum available}\\ \text{of a currency (input)}\end{pmatrix} = \begin{pmatrix}\text{Total sum converted to}\\ \text{other currencies (output)}\end{pmatrix}$$

1. *Dollar* ($i = 1$):

$$\begin{aligned}\text{Total available dollars} &= \text{Initial dollar amount} + \\ &\quad \text{dollar amount from other currencies} \\ &= I + (€ \rightarrow \$) + (£ \rightarrow \$) + (¥ \rightarrow \$) + (KD \rightarrow \$) \\ &= I + \tfrac{1}{.769}x_{21} + \tfrac{1}{.625}x_{31} + \tfrac{1}{105}x_{41} + \tfrac{1}{.342}x_{51}\end{aligned}$$

$$\begin{aligned}\text{Total distributed dollars} &= \text{Final dollar holdings} + \\ &\quad \text{dollar amount to other currencies} \\ &= y + (\$ \rightarrow €) + (\$ \rightarrow £) + (\$ \rightarrow ¥) + (\$ \rightarrow KD) \\ &= y + x_{12} + x_{13} + x_{14} + x_{15}\end{aligned}$$

Given $I = 5$, the dollar constraint thus becomes

$$y + x_{12} + x_{13} + x_{14} + x_{15} - \left(\tfrac{1}{.769}x_{21} + \tfrac{1}{.625}x_{31} + \tfrac{1}{105}x_{41} + \tfrac{1}{.342}x_{51}\right) = 5$$

2. *Euro* ($i = 2$):

$$\begin{aligned}\text{Total available euros} &= (\$ \rightarrow €) + (£ \rightarrow €) + (¥ \rightarrow €) + (KD \rightarrow €) \\ &= .769x_{12} + \tfrac{1}{.813}x_{32} + \tfrac{1}{137}x_{42} + \tfrac{1}{.445}x_{52}\end{aligned}$$

$$\begin{aligned}\text{Total distributed euros} &= (€ \rightarrow \$) + (€ \rightarrow £) + (€ \rightarrow ¥) + (€ \rightarrow KD) \\ &= x_{21} + x_{23} + x_{24} + x_{25}\end{aligned}$$

Thus, the constraint is

$$x_{21} + x_{23} + x_{24} + x_{25} - \left(.769x_{12} + \tfrac{1}{.813}x_{32} + \tfrac{1}{137}x_{42} + \tfrac{1}{.445}x_{52}\right) = 0$$

3. *Pound* ($i = 3$):

$$\text{Total available pounds} = (\$ \rightarrow £) + (€ \rightarrow £) + (¥ \rightarrow £) + (KD \rightarrow £)$$
$$= .625x_{13} + .813x_{23} + \tfrac{1}{169}x_{43} + \tfrac{1}{.543}x_{53}$$

$$\text{Total distributed pounds} = (£ \rightarrow \$) + (£ \rightarrow €) + (£ \rightarrow ¥) + (£ \rightarrow KD)$$
$$= x_{31} + x_{32} + x_{34} + x_{35}$$

Thus, the constraint is

$$x_{31} + x_{32} + x_{34} + x_{35} - .625x_{13} + .813x_{23} + \tfrac{1}{169}x_{43} + \tfrac{1}{.543}x_{53} = 0$$

4. *Yen* ($i = 4$):

$$\text{Total available yen} = (\$ \rightarrow ¥) + (€ \rightarrow ¥) + (£ \rightarrow ¥) + (KD \rightarrow ¥)$$
$$= 105x_{14} + 137x_{24} + 169x_{34} + \tfrac{1}{.0032}x_{54}$$

$$\text{Total distributed yen} = (¥ \rightarrow \$) + (¥ \rightarrow €) + (¥ \rightarrow £) + (¥ \rightarrow KD)$$
$$= x_{41} + x_{42} + x_{43} + x_{45}$$

Thus, the constraint is

$$x_{41} + x_{42} + x_{43} + x_{45} - \left(105x_{14} + 137x_{24} + 169x_{34} + \tfrac{1}{.0032}x_{54}\right) = 0$$

5. *KD* ($i = 5$):

$$\text{Total available KDs} = (KD \rightarrow \$) + (KD \rightarrow €) + (KD \rightarrow £) + (KD \rightarrow ¥)$$
$$= .342x_{15} + .445x_{25} + .543x_{35} + .0032x_{45}$$

$$\text{Total distributed KDs} = (\$ \rightarrow KD) + (€ \rightarrow KD) + (£ \rightarrow KD) + (¥ \rightarrow KD)$$
$$= x_{51} + x_{52} + x_{53} + x_{54}$$

Thus, the constraint is

$$x_{51} + x_{52} + x_{53} + x_{54} - (.342x_{15} + .445x_{25} + .543x_{35} + .0032x_{45}) = 0$$

The only remaining constraints are the transaction limits, which are 5 million dollars, 3 million euros, 3.5 million pounds, 100 million yen, and 2.8 million KDs. These can be translated as

$$x_{1j} \le 5, j = 2, 3, 4, 5$$
$$x_{2j} \le 3, j = 1, 3, 4, 5$$
$$x_{3j} \le 3.5, j = 1, 2, 4, 5$$
$$x_{4j} \le 100, j = 1, 2, 3, 5$$
$$x_{5j} \le 2.8, j = 1, 2, 3, 4$$

The complete model is now given as

$$\text{Maximize } z = y$$

subject to

$$y + x_{12} + x_{13} + x_{14} + x_{15} - \left(\tfrac{1}{.769}x_{21} + \tfrac{1}{.625}x_{31} + \tfrac{1}{105}x_{41} + \tfrac{1}{.342}x_{51}\right) = 5$$

$$x_{21} + x_{23} + x_{24} + x_{25} - \left(.769x_{12} + \tfrac{1}{.813}x_{32} + \tfrac{1}{137}x_{42} + \tfrac{1}{.445}x_{52}\right) = 0$$

$$x_{31} + x_{32} + x_{34} + x_{35} - \left(.625x_{13} + .813x_{23} + \tfrac{1}{169}x_{43} + \tfrac{1}{.543}x_{53}\right) = 0$$

$$x_{41} + x_{42} + x_{43} + x_{45} - \left(105x_{14} + 137x_{24} + 169x_{34} + \tfrac{1}{.0032}x_{54}\right) = 0$$

$$x_{51} + x_{52} + x_{53} + x_{54} - (.342x_{15} + .445x_{25} + .543x_{35} + .0032x_{45}) = 0$$

$$x_{1j} \le 5, j = 2, 3, 4, 5$$

$$x_{2j} \le 3, j = 1, 3, 4, 5$$

$$x_{3j} \le 3.5, j = 1, 2, 4, 5$$

$$x_{4j} \le 100, j = 1, 2, 3, 5$$

$$x_{5j} \le 2.8, j = 1, 2, 3, 4$$

$$x_{ij} \ge 0, \text{ for all } i \text{ and } j$$

Solution:

The optimum solution (using file amplEx2.3-2.txt or solverEx2.3-2.xls) is:

Solution	Interpretation
$y = 5.09032$	Final holdings = \$5,090,320. Net dollar gain = \$90,320, which represents a 1.8064% rate of return
$x_{12} = 1.46206$	Buy \$1,462,060 worth of euros
$x_{15} = 5$	Buy \$5,000,000 worth of KD
$x_{25} = 3$	Buy €3,000,000 worth of KD
$x_{31} = 3.5$	Buy £3,500,000 worth of dollars
$x_{32} = 0.931495$	Buy £931,495 worth of euros
$x_{41} = 100$	Buy ¥100,000,000 worth of dollars
$x_{42} = 100$	Buy ¥100,000,000 worth of euros
$x_{43} = 100$	Buy ¥100,000,000 worth of pounds
$x_{53} = 2.085$	Buy KD2,085,000 worth of pounds
$x_{54} = .96$	Buy KD960,000 worth of yen

Remarks. At first it may appear that the solution is nonsensical because it calls for using $x_{12} + x_{15} = 1.46206 + 5 = 6.46206$, or \$6,462,060 to buy euros and KDs when the initial dollar amount is only \$5,000,000. Where do the extra dollars come from? What happens in practice is that the given solution is submitted to the currency dealer as *one* order, meaning we do not wait until we accumulate enough currency of a certain type before making a buy. In the end, the net result of all these transactions is a net cost of $5,000,000 to the investor. This can be seen by summing up all the dollar transactions in the solution:

$$I = y + x_{12} + x_{13} + x_{14} + x_{15} - \left(\tfrac{1}{.769}x_{21} + \tfrac{1}{.625}x_{31} + \tfrac{1}{105}x_{41} + \tfrac{1}{.342}x_{51}\right)$$

$$= 5.09032 + 1.46206 + 5 - \left(\tfrac{3.5}{.625} + \tfrac{100}{105}\right) = 5$$

Notice that x_{21}, x_{31}, x_{41} and x_{51} are in euro, pound, yen, and KD, respectively, and hence must be converted to dollars.

PROBLEM SET 2.3B

1. Modify the arbitrage model to account for a commission that amounts to .1% of any currency buy. Assume that the commission does not affect the circulating funds and that it is collected after the entire order is executed. How does the solution compare with that of the original model?
*2. Suppose that the company is willing to convert the initial $5 million to any other currency that will provide the highest rate of return. Modify the original model to determine which currency is the best.
3. Suppose the initial amount I = $7 million and that the company wants to convert it optimally to a combination of euros, pounds, and yen. The final mix may not include more than €2 million, £3 million, and ¥200 million. Modify the original model to determine the optimal buying mix of the three currencies.
4. Suppose that the company wishes to buy $6 million. The transaction limits for different currencies are the same as in the original problem. Devise a buying schedule for this transaction, given that mix may not include more than €3 million, £2 million, and KD2 million.
5. Suppose that the company has $2 million, €5 million, £4 million. Devise a buy-sell order that will improve the overall holdings converted to yen.

2.3.3 Investment

Today's investors are presented with multitudes of investment opportunities. Examples of investment problems are capital budgeting for projects, bond investment strategy, stock portfolio selection, and establishment of bank loan policy. In many of these situations, linear programming can be used to select the optimal mix of opportunities that will maximize return while meeting the investment conditions set by the investor.

Example 2.3-3 (Loan Policy Model)

Thriftem Bank is in the process of devising a loan policy that involves a maximum of $12 million. The following table provides the pertinent data about available types of loans.

Type of loan	Interest rate	Bad-debt ratio
Personal	.140	.10
Car	.130	.07
Home	.120	.03
Farm	.125	.05
Commercial	.100	.02

Bad debts are unrecoverable and produce no interest revenue.

Competition with other financial institutions requires that the bank allocate at least 40% of the funds to farm and commercial loans. To assist the housing industry in the region, home loans must equal at least 50% of the personal, car, and home loans. The bank also has a stated policy of not allowing the overall ratio of bad debts on all loans to exceed 4%.

Mathematical Model: The situation seeks to determine the amount of loan in each category, thus leading to the following definitions of the variables:

$$x_1 = \text{personal loans (in millions of dollars)}$$
$$x_2 = \text{car loans}$$
$$x_3 = \text{home loans}$$
$$x_4 = \text{farm loans}$$
$$x_5 = \text{commercial loans}$$

The objective of the Thriftem Bank is to maximize its net return, the difference between interest revenue and lost bad debts. The interest revenue is accrued only on loans in good standing. Thus, because 10% of personal loans are lost to bad debt, the bank will receive interest on only 90% of the loan—that is, it will receive 14% interest on $.9x_1$ of the original loan x_1. The same reasoning applies to the remaining four types of loans. Thus,

$$\begin{aligned}\text{Total interest} &= .14(.9x_1) + .13(.93x_2) + .12(.97x_3) + .125(.95x_4) + .1(.98x_5)\\ &= .126x_1 + .1209x_2 + .1164x_3 + .11875x_4 + .098x_5\end{aligned}$$

We also have

$$\text{Bad debt} = .1x_1 + .07x_2 + .03x_3 + .05x_4 + .02x_5$$

The objective function is thus expressed as

$$\begin{aligned}\text{Maximize } z &= \text{Total interest—Bad debt}\\ &= (.126x_1 + .1209x_2 + .1164x_3 + .11875x_4 + .098x_5)\\ &\quad - (.1x_1 + .07x_2 + .03x_3 + .05x_4 + .02x_5)\\ &= .026x_1 + .0509x_2 + .0864x_3 + .06875x_4 + .078x_5\end{aligned}$$

The problem has five constraints:

1. *Total funds should not exceed \$12 (million):*

$$x_1 + x_2 + x_3 + x_4 + x_5 \le 12$$

2. *Farm and commercial loans equal at least 40% of all loans:*

$$x_4 + x_5 \ge .4(x_1 + x_2 + x_3 + x_4 + x_5)$$

or

$$.4x_1 + .4x_2 + .4x_3 - .6x_4 - .6x_5 \le 0$$

3. *Home loans should equal at least 50% of personal, car, and home loans:*

$$x_3 \geq .5(x_1 + x_2 + x_3)$$

or

$$.5x_1 + .5x_2 - .5x_3 \leq 0$$

4. *Bad debts should not exceed 4% of all loans:*

$$.1x_1 + .07x_2 + .03x_3 + .05x_4 + .02x_5 \leq .04(x_1 + x_2 + x_3 + x_4 + x_5)$$

or

$$.06x_1 + .03x_2 - .01x_3 + .01x_4 - .02x_5 \leq 0$$

5. *Nonnegativity*:

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$$

A subtle assumption in the preceding formulation is that all loans are issued at approximately the same time. This assumption allows us to ignore differences in the time value of the funds allocated to the different loans.

Solution:

The optimal solution is

$$z = .99648, x_1 = 0, x_2 = 0, x_3 = 7.2, x_4 = 0, x_5 = 4.8$$

Remarks.

1. You may be wondering why we did not define the right-hand side of the second constraint as $.4 \times 12$ instead of $.4(x_1 + x_2 + x_3 + x_4 + x_5)$. After all, it seems logical that the bank would want to loan out all \$12 (million). The answer is that the second usage does not "rob" the model of this possibility. If the optimum solution needs all \$12 (million), the given constraint will allow it. But there are two important reasons why you should not use $.4 \times 12$: (1) If other constraints in the model are such that all \$12 (million) *cannot* be used (for example, the bank may set caps on the different loans), then the choice $.4 \times 12$ could lead to an infeasible or incorrect solution. (2) If you want to experiment with the effect of changing available funds (say from \$12 to \$13 million) on the optimum solution, there is a real chance that you may forget to change $.4 \times 12$ to $.4 \times 13$, in which case the solution you get will not be correct. A similar reasoning applies to the left-hand side of the fourth constraint.
2. The optimal solution calls for allocating all \$12 million: \$7.2 million to home loans and \$4.8 million to commercial loans. The remaining categories receive none. The return on the investment is computed as

$$\text{Rate of return} = \frac{z}{12} = \frac{.99648}{12} = .08034$$

This shows that the combined annual rate of return is 8.034%, which is less than the best *net* interest rate (=.0864 for home loans), and one wonders why the optimum does not take advantage of this opportunity. The answer is that the restriction stipulating that farm and commercial loans account for at least 40% of all loans (constraint 2) forces the solution to allocate \$4.8 million to commercial loans at the lower *net* rate of .078, hence lowering the overall interest rate to $\frac{.0864 \times 7.2 + .078 \times 4.8}{12} = .08034$. In fact, if we remove constraint 2, the optimum will allocate all the funds to home loans at the higher 8.64% rate.

PROBLEM SET 2.3C

1. Fox Enterprises is considering six projects for possible construction over the next four years. The expected (present value) returns and cash outlays for the projects are given below. Fox can undertake any of the projects partially or completely. A partial undertaking of a project will prorate both the return and cash outlays proportionately.

	Cash outlay ($1000)				
Project	*Year 1*	*Year 2*	*Year 3*	*Year 4*	Return ($1000)
1	10.5	14.4	2.2	2.4	32.40
2	8.3	12.6	9.5	3.1	35.80
3	10.2	14.2	5.6	4.2	17.75
4	7.2	10.5	7.5	5.0	14.80
5	12.3	10.1	8.3	6.3	18.20
6	9.2	7.8	6.9	5.1	12.35
Available funds ($1000)	60.0	70.0	35.0	20.0	

(a) Formulate the problem as a linear program, and determine the optimal project mix that maximizes the total return. Ignore the time value of money.

(b) Suppose that if a portion of project 2 is undertaken then at least an equal portion of project 6 must undertaken. Modify the formulation of the model and find the new optimal solution.

(c) In the original model, suppose that any funds left at the end of a year are used in the next year. Find the new optimal solution, and determine how much each year "borrows" from the preceding year. For simplicity, ignore the time value of money.

(d) Suppose in the original model that the yearly funds available for any year can be exceeded, if necessary, by borrowing from other financial activities within the company. Ignoring the time value of money, reformulate the LP model, and find the optimum solution. Would the new solution require borrowing in any year? If so, what is the rate of return on borrowed money?

*2. Investor Doe has $10,000 to invest in four projects. The following table gives the cash flow for the four investments.

	Cash flow ($1000) at the start of				
Project	*Year 1*	*Year 2*	*Year 3*	*Year 4*	*Year 5*
1	−1.00	0.50	0.30	1.80	1.20
2	−1.00	0.60	0.20	1.50	1.30
3	0.00	−1.00	0.80	1.90	0.80
4	−1.00	0.40	0.60	1.80	0.95

The information in the table can be interpreted as follows: For project 1, $1.00 invested at the start of year 1 will yield $.50 at the start of year 2, $.30 at the start of year 3, $1.80 at the start of year 4, and $1.20 at the start of year 5. The remaining entries can be interpreted similarly. The entry 0.00 indicates that no transaction is taking place. Doe has the additional option of investing in a bank account that earns 6.5% annually. All funds accumulated at the end of one year can be reinvested in the following year. Formulate the problem as a linear program to determine the optimal allocation of funds to investment opportunities.

3. HiRise Construction can bid on two 1-year projects. The following table provides the quarterly cash flow (in millions of dollars) for the two projects.

	Cash flow (in millions of $) at				
Project	*1/1/08*	*4/1/08*	*7/1/08*	*10/1/08*	*12/31/08*
I	−1.0	−3.1	−1.5	1.8	5.0
II	−3.0	−2.5	1.5	1.8	2.8

HiRise has cash funds of $1 million at the beginning of each quarter and may borrow at most $1 million at a 10% nominal annual interest rate. Any borrowed money must be returned at the end of the quarter. Surplus cash can earn quarterly interest at an 8% nominal annual rate. Net accumulation at the end of one quarter is invested in the next quarter.

(a) Assume that HiRise is allowed partial or full participation in the two projects. Determine the level of participation that will maximize the net cash accumulated on 12/31/2008.

(b) Is it possible in any quarter to borrow money and simultaneously end up with surplus funds? Explain.

4. In anticipation of the immense college expenses, a couple have started an annual investment program on their child's eighth birthday that will last until the eighteenth birthday. The couple estimate that they will be able to invest the following amounts at the beginning of each year:

Year	1	2	3	4	5	6	7	8	9	10
Amount ($)	2000	2000	2500	2500	3000	3500	3500	4000	4000	5000

To avoid unpleasant surprises, they want to invest the money safely in the following options: Insured savings with 7.5% annual yield, six-year government bonds that yield 7.9% and have a current market price equal to 98% of face value, and nine-year municipal bonds yielding 8.5% and having a current market price of 1.02 of face value. How should the couple invest the money?

***5.** A business executive has the option to invest money in two plans: Plan A guarantees that each dollar invested will earn $.70 a year later, and plan B guarantees that each dollar invested will earn $2 after 2 years. In plan A, investments can be made annually, and in plan B, investments are allowed for periods that are multiples of two years only. How should the executive invest $100,000 to maximize the earnings at the end of 3 years?

6. A gambler plays a game that requires dividing bet money among four choices. The game has three outcomes. The following table gives the corresponding gain or loss per dollar for the different options of the game.

	Return per dollar deposited in choice			
Outcome	*1*	*2*	*3*	*4*
1	−3	4	−7	15
2	5	−3	9	4
3	3	−9	10	−8

The gambler has a total of $500, which may be played only once. The exact outcome of the game is not known a priori. Because of this uncertainty, the gambler's strategy is to maximize the *minimum* return produced by the three outcomes. How should the gambler allocate the $500 among the four choices? (*Hint:* The gambler's net return may be positive, zero, or negative.)

7. (Lewis, 1996) Monthly bills in a household are received monthly (e.g., utilities and home mortgage), quarterly (e.g., estimated tax payment), semiannually (e.g., insurance), or annually (e.g., subscription renewals and dues). The following table provides the monthly bills for next year.

Month	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.	Total
$	800	1200	400	700	600	900	1500	1000	900	1100	1300	1600	12000

To account for these expenses, the family sets aside $1000 per month, which is the average of the total divided by 12 months. If the money is deposited in a regular savings account, it can earn 4% annual interest, provided it stays in the account at least one month. The bank also offers 3-month and 6-month certificates of deposit that can earn 5.5% and 7% annual interest, respectively. Develop a 12-month investment schedule that will maximize the family's total return for the year. State any assumptions or requirements needed to reach a feasible solution.

2.3.4 Production Planning and Inventory Control

There is a wealth of LP applications to production and inventory control, ranging from simple allocation of machining capacity to meet demand to the more complex case of using inventory to "dampen" the effect of erratic change in demand over a given planning horizon and of using hiring and firing to respond to changes in workforce needs. This section presents three examples. The first deals with the scheduling of products using common production facilities to meet demand during a single period, the second deals with the use of inventory in a multiperiod production system to fill future demand, and the third deals with the use of a combined inventory and worker hiring/firing to "smooth" production over a multiperiod planning horizon with fluctuating demand.

Example 2.3-4 (Single-Period Production Model)

In preparation for the winter season, a clothing company is manufacturing parka and goose overcoats, insulated pants, and gloves. All products are manufactured in four different departments: cutting, insulating, sewing, and packaging. The company has received firm orders for its products. The contract stipulates a penalty for undelivered items. The following table provides the pertinent data of the situation.

	Time per units (hr)				
Department	*Parka*	*Goose*	*Pants*	*Gloves*	Capacity (hr)
Cutting	.30	.30	.25	.15	1000
Insulating	.25	.35	.30	.10	1000
Sewing	.45	.50	.40	.22	1000
Packaging	.15	.15	.1	.05	1000
Demand	800	750	600	500	
Unit profit	$30	$40	$20	$10	
Unit penalty	$15	$20	$10	$8	

Devise an optimal production plan for the company.

Mathematical Model: The definition of the variables is straightforward. Let

$$x_1 = \text{number of parka jackets}$$
$$x_2 = \text{number of goose jackets}$$
$$x_3 = \text{number of pairs of pants}$$
$$x_4 = \text{number of pairs of gloves}$$

The company is penalized for not meeting demand. This means that the objective of the problem is to maximize the net receipts, defined as

$$\text{Net receipts} = \text{Total profit} - \text{Total penalty}$$

The total profit is readily expressed as $30x_1 + 40x_2 + 20x_3 + 10x_4$. The total penalty is a function of the shortage quantities (= demand − units supplied of each product). These quantities can be determined from the following demand limits:

$$x_1 \leq 800, x_2 \leq 750, x_3 \leq 600, x_4 \leq 500$$

A demand is not fulfilled if its constraint is satisfied as a strict inequality. For example, if 650 parka jackets are produced, then $x_1 = 650$, which leads to a shortage of $800 - 650 = 150$ parka jackets. We can express the shortage of any product algebraically by defining a new nonnegative variable—namely,

$$s_j = \text{Number of shortage units of product } j, j = 1, 2, 3, 4$$

In this case, the demand constraints can be written as

$$x_1 + s_1 = 800, x_2 + s_2 = 750, x_3 + s_3 = 600, x_4 + s_4 = 500$$
$$x_j \geq 0, s_j \geq 0, j = 1, 2, 3, 4$$

We can now compute the shortage penalty as $15s_1 + 20s_2 + 10s_3 + 8s_4$. Thus, the objective function can be written as

$$\text{Maximize } z = 30x_1 + 40x_2 + 20x_3 + 10x_4 - (15s_1 + 20s_2 + 10s_3 + 8s_4)$$

To complete the model, the remaining constraints deal with the production capacity restrictions; namely

$$.30x_1 + .30x_2 + .25x_3 + .15x_4 \leq 1000 \quad \text{(Cutting)}$$
$$.25x_1 + .35x_2 + .30x_3 + .10x_4 \leq 1000 \quad \text{(Insulating)}$$
$$.45x_1 + .50x_2 + .40x_3 + .22x_4 \leq 1000 \quad \text{(Sewing)}$$
$$.15x_1 + .15x_2 + .10x_3 + .05x_4 \leq 1000 \quad \text{(Packaging)}$$

The complete model thus becomes

$$\text{Maximize } z = 30x_1 + 40x_2 + 20x_3 + 10x_4 - (15s_1 + 20s_2 + 10s_3 + 8s_4)$$

subject to

$$.30x_1 + .30x_2 + .25x_3 + .15x_4 \le 1000$$
$$.25x_1 + .35x_2 + .30x_3 + .10x_4 \le 1000$$
$$.45x_1 + .50x_2 + .40x_3 + .22x_4 \le 1000$$
$$.15x_1 + .15x_2 + .10x_3 + .05x_4 \le 1000$$
$$x_1 + s_1 = 800, x_2 + s_2 = 750, x_3 + s_3 = 600, x_4 + s_4 = 500$$
$$x_j \ge 0, s_j \ge 0, j = 1, 2, 3, 4$$

Solution:

The optimum solution is $z = \$64{,}625, x_1 = 850, x_2 = 750,\ x_3 = 387.5,\ x_4 = 500,\ s_1 = s_2 = s_4 = 0, s_3 = 212.5$. The solution satisfies all the demand for both types of jackets and the gloves. A shortage of 213 (rounded up from 212.5) pairs of pants will result in a penalty cost of $213 \times \$10 = \$2130$.

Example 2.3-5 (Multiple Period Production-Inventory Model)

Acme Manufacturing Company has contracted to deliver home windows over the next 6 months. The demands for each month are 100, 250, 190, 140, 220, and 110 units, respectively. Production cost per window varies from month to month depending on the cost of labor, material, and utilities. Acme estimates the production cost per window over the next 6 months to be \$50, \$45, \$55, \$48, \$52, and \$50, respectively. To take advantage of the fluctuations in manufacturing cost, Acme may elect to produce more than is needed in a given month and hold the excess units for delivery in later months. This, however, will incur storage costs at the rate of \$8 per window per month assessed on end-of-month inventory. Develop a linear program to determine the optimum production schedule.

Mathematical Model: The variables of the problem include the monthly production amount and the end-of-month inventory. For $i = 1, 2, \ldots, 6$, let

$$x_i = \text{Number of units produced in month } i$$
$$I_i = \text{Inventory units left at the end of month } i$$

The relationship between these variables and the monthly demand over the six-month horizon is represented by the schematic diagram in Figure 2.5. The system starts empty, which means that $I_0 = 0$.

The objective function seeks to minimize the sum of the production and end-of-month inventory costs. Here we have,

$$\text{Total production cost} = 50x_1 + 45x_2 + 55x_3 + 48x_4 + 52x_5 + 50x_6$$
$$\text{Total inventory cost} = 8(I_1 + I_2 + I_3 + I_4 + I_5 + I_6)$$

Thus the objective function is

$$\text{Minimize } z = 50x_1 + 45x_2 + 55x_3 + 48x_4 + 52x_5 + 50x_6$$
$$+ 8(I_1 + I_2 + I_3 + I_4 + I_5 + I_6)$$

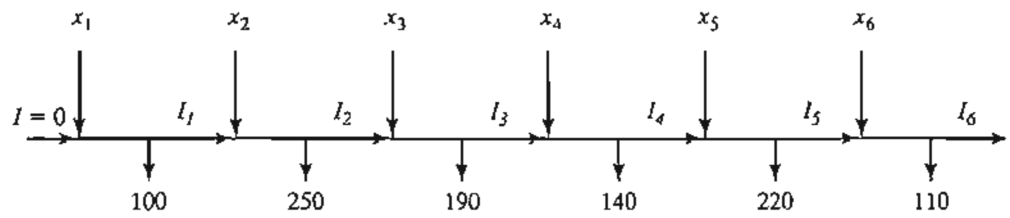


FIGURE 2.5
Schematic representation of the production-inventory system

The constraints of the problem can be determined directly from the representation in Figure 2.5. For each period we have the following balance equation:

$$\text{Beginning inventory} + \text{Production amount} - \text{Ending inventory} = \text{Demand}$$

This is translated mathematically for the individual months as

$$\begin{aligned}
I_0 + x_1 - I_1 &= 100 \quad \text{(Month 1)}\\
I_1 + x_2 - I_2 &= 250 \quad \text{(Month 2)}\\
I_2 + x_3 - I_3 &= 190 \quad \text{(Month 3)}\\
I_3 + x_4 - I_4 &= 140 \quad \text{(Month 4)}\\
I_4 + x_5 - I_5 &= 220 \quad \text{(Month 5)}\\
I_5 + x_6 - I_6 &= 110 \quad \text{(Month 6)}\\
x_i, I_i &\geq 0, \text{ for all } i = 1, 2, \ldots, 6\\
I_0 &= 0
\end{aligned}$$

For the problem, $I_0 = 0$ because the situation starts with no initial inventory. Also, in any optimal solution, the ending inventory I_6 will be zero, because it is not logical to end the horizon with positive inventory, which can only incur additional inventory cost without serving any purpose.

The complete model is now given as

$$\text{Minimize } z = 50x_1 + 45x_2 + 55x_3 + 48x_4 + 52x_5 + 50x_6 + 8(I_1 + I_2 + I_3 + Ix_4 + I_5 + I_6)$$

subject to

$$\begin{aligned}
x_1 - I_1 &= 100 \quad \text{(Month 1)}\\
I_1 + x_2 - I_2 &= 250 \quad \text{(Month 2)}\\
I_2 + x_3 - I_3 &= 190 \quad \text{(Month 3)}\\
I_3 + x_4 - I_4 &= 140 \quad \text{(Month 4)}\\
I_4 + x_5 - I_5 &= 220 \quad \text{(Month 5)}\\
I_5 + x_6 - I_6 &= 110 \quad \text{(Month 6)}\\
x_i, I_i &\geq 0, \text{ for all } i = 1, 2, \ldots, 6
\end{aligned}$$

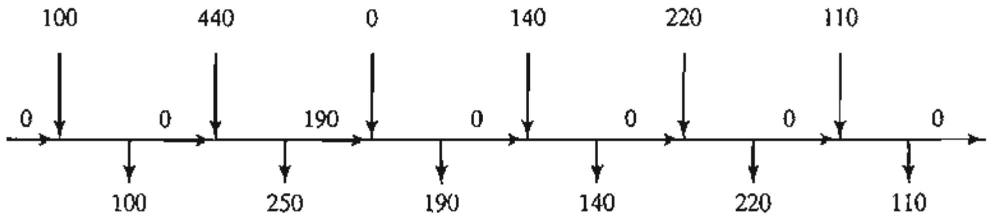


FIGURE 2.6
Optimum solution of the production-inventory problem

Solution:

The optimum solution is summarized in Figure 2.6. It shows that each month's demand is satisfied directly from the month's production, except for month 2 whose production quantity of 440 units covers the demand for both months 2 and 3. The total associated cost is $z = \$49,980$.

Example 2.3-6 (Multiperiod Production Smoothing Model)

A company will manufacture a product for the next four months: March, April, May, and June. The demands for each month are 520, 720, 520, and 620 units, respectively. The company has a steady workforce of 10 employees but can meet fluctuating production needs by hiring and firing temporary workers, if necessary. The extra costs of hiring and firing in any month are \$200 and \$400 per worker, respectively. A permanent worker can produce 12 units per month, and a temporary worker, lacking comparable experience, only produce 10 units per month. The company can produce more than needed in any month and carry the surplus over to a succeeding month at a holding cost of \$50 per unit per month. Develop an optimal hiring/firing policy for the company over the four-month planning horizon.

Mathematical Model: This model is similar to that of Example 2.3-5 in the general sense that each month has its production, demand, and ending inventory. There are two exceptions: (1) accounting for the permanent versus the temporary workforce, and (2) accounting for the cost of hiring and firing in each month.

Because the permanent 10 workers cannot be fired, their impact can be accounted for by subtracting the units they produce from the respective monthly demand. The remaining demand, if any, is satisfied through hiring and firing of temps. From the standpoint of the model, the net demand for each month is

$$\text{Demand for March} = 520 - 12 \times 10 = 400 \text{ units}$$
$$\text{Demand for April} = 720 - 12 \times 10 = 600 \text{ units}$$
$$\text{Demand for May} = 520 - 12 \times 10 = 400 \text{ units}$$
$$\text{Demand for June} = 620 - 12 \times 10 = 500 \text{ units}$$

For $i = 1, 2, 3, 4$, the variables of the model can be defined as

x_i = Net number of temps at the start of month i *after* any hiring or firing

S_i = Number of temps hired or fired at the start of month i

I_i = Units of ending inventory for month i

The variables x_i and I_i, by definition, must assume nonnegative values. On the other hand, the variable S_i can be positive when new temps are hired, negative when workers are fired, and zero if no hiring or firing occurs. As a result, the variable must be *unrestricted in sign*. This is the first instance in this chapter of using an unrestricted variable. As we will see shortly, special substitution is needed to allow the implementation of hiring and firing in the model.

The objective is to minimize the sum of the cost of hiring and firing plus the cost of holding inventory from one month to the next. The treatment of the inventory cost is similar to the one given in Example 2.3-5—namely,

$$\text{Inventory holding cost} = 50(I_1 + I_2 + I_3)$$

(Note that $I_4 = 0$ in the optimum solution.) The cost of hiring and firing is a bit more involved. We know that in any optimum solution, at least 40 temps $\left(= \frac{400}{10}\right)$ must be hired at the start of March to meet the month's demand. However, rather than treating this situation as a special case, we can let the optimization process take care of it automatically. Thus, given that the costs of hiring and firing a temp are \$200 and \$400, respectively, we have

$$\begin{pmatrix}\text{Cost of hiring}\\ \text{and firing}\end{pmatrix} = 200\begin{pmatrix}\text{Number of hired temps}\\ \text{at the start of}\\ \text{March, April, May, and June}\end{pmatrix}$$

$$+ 400\begin{pmatrix}\text{Number of fired temps}\\ \text{at the start of}\\ \text{March, April, May, and June}\end{pmatrix}$$

To translate this equation mathematically, we will need to develop the constraints first.

The constraints of the model deal with inventory and hiring and firing. First we develop the inventory constraints. Defining x_i as the number of temps available in month i and given that the productivity of a temp is 10 units per month, the number of units produced in the same month is $10x_i$. Thus the inventory constraints are

$$\begin{aligned}
10x_1 &= 400 + I_1 \quad &\text{(March)}\\
I_1 + 10x_2 &= 600 + I_2 \quad &\text{(April)}\\
I_2 + 10x_3 &= 400 + I_3 \quad &\text{(May)}\\
I_3 + 10x_4 &= 500 \quad &\text{(June)}\\
x_1, x_2, x_3, x_4 &\geq 0, I_1, I_2, I_3 \geq 0
\end{aligned}$$

Next, we develop the constraints dealing with hiring and firing. First, note that the temp workforce starts with x_1 workers at the beginning of March. At the start of April, x_1 will be adjusted (up or down) by S_2 to generate x_2. The same idea applies to x_3 and x_4. These observations lead to the following equations

$$\begin{aligned}
x_1 &= S_1\\
x_2 &= x_1 + S_2\\
x_3 &= x_2 + S_3
\end{aligned}$$

$$x_4 = x_3 + S_4$$

$$S_1, S_2, S_3, S_4 \text{ unrestricted in sign}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

The variables S_1, S_2, S_3, and S_4 represent hiring when they are strictly positive and firing when they are strictly negative. However, this "qualitative" information cannot be used in a mathematical expression. Instead, we use the following substitution:

$$S_i = S_i^- - S_i^+, \text{ where } S_i^-, S_i^+ \geq 0$$

The unrestricted variable S_i is now the difference between two nonnegative variables S_i^- and S_i^+. We can think of S_i^- as the number of temps hired and S_i^+ as the number of temps fired. For example, if $S_i^- = 5$ and $S_i^+ = 0$ then $S_i = 5 - 0 = +5$, which represents hiring. If $S_i^- = 0$ and $S_i^+ = 7$ then $S_i = 0 - 7 = -7$, which represents firing. In the first case, the corresponding cost of hiring is $200S_i^- = 200 \times 5 = \$1000$ and in the second case the corresponding cost of firing is $400S_i^+ = 400 \times 7 = \$2800$. This idea is the basis for the development of the objective function.

First we need to address an important point: What if both S_i^- and S_i^+ are positive? The answer is that this cannot happen because it implies that the solution calls for both hiring and firing in the same month. Interestingly, the theory of linear programming (see Chapter 7) tells us that S_i^- and S_i^+ can never be positive simultaneously, a result that confirms intuition.

We can now write the cost of hiring and firing as follows:

$$\text{Cost of hiring} = 200(S_1^- + S_2^- + S_3^- + S_4^-)$$

$$\text{Cost of firing} = 400(S_1^+ + S_2^+ + S_3^+ + S_4^+)$$

The complete model is

$$\text{Minimize } z = 50(I_1 + I_2 + I_3 + I_4) + 200(S_1^- + S_2^- + S_3^- + S_4^-) + 400(S_1^+ + S_2^+ + S_3^+ + S_4^+)$$

subject to

$$10x_1 = 400 + I_1$$

$$I_1 + 10x_2 = 600 + I_2$$

$$I_2 + 10x_3 = 400 + I_3$$

$$I_3 + 10x_4 = 500$$

$$x_1 = S_1^- - S_1^+$$

$$x_2 = x_1 + S_2^- - S_2^+$$

$$x_3 = x_2 + S_3^- - S_3^+$$

$$x_4 = x_3 + S_4^- - S_4^+$$

$$S_1^-, S_1^+, S_2^-, S_2^+, S_3^-, S_3^+, S_4^-, S_4^+ \geq 0$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$I_1, I_2, I_3 \geq 0$$

Solution:

The optimum solution is $z = \$19{,}500$, $x_1 = 50, x_2 = 50$, $x_3 = 45, x_4 = 45$, $S_1^- = 50$, $S_3^+ = 5$, $I_1 = 100$, $I_3 = 50$. All the remaining variables are zero. The solution calls for hiring 50 temps in March ($S_1^- = 50$) and holding the workforce steady till May, when 5 temps are fired ($S_3^+ = 5$). No further hiring or firing is recommended until the end of June, when, presumably, all temps are terminated. This solution requires 100 units of inventory to be carried into May and 50 units to be carried into June.

PROBLEM SET 2.3D

1. Toolco has contracted with AutoMate to supply their automotive discount stores with wrenches and chisels. AutoMate's weekly demand consists of at least 1500 wrenches and 1200 chisels. Toolco cannot produce all the requested units with its present one-shift capacity and must use overtime and possibly subcontract with other tool shops. The result is an increase in the production cost per unit, as shown in the following table. Market demand restricts the ratio of chisels to wrenches to at least 2:1.

Tool	Production type	Weekly production range (units)	Unit cost ($)
Wrenches	Regular	0–550	2.00
	Overtime	551–800	2.80
	Subcontracting	$801-\infty$	3.00
Chisel	Regular	0–620	2.10
	Overtime	621–900	3.20
	Subcontracting	$901-\infty$	4.20

   (a) Formulate the problem as a linear program, and determine the optimum production schedule for each tool.

   (b) Relate the fact that the production cost function has increasing unit costs to the validity of the model.

2. Four products are processed sequentially on three machines. The following table gives the pertinent data of the problem.

		Manufacturing time (hr) per unit				
Machine	Cost per hr ($)	*Product 1*	*Product 2*	*Product 3*	*Product 4*	Capacity (hr)
1	10	2	3	4	2	500
2	5	3	2	1	2	380
3	4	7	3	2	1	450
Unit selling price ($)		75	70	55	45	

Formulate the problem as an LP model, and find the optimum solution.

*3. A manufacturer produces three models, I, II, and III, of a certain product using raw materials A and B. The following table gives the data for the problem:

	Requirements per unit			
Raw material	*I*	*II*	*III*	Availability
A	2	3	5	4000
B	4	2	7	6000
Minimum demand	200	200	150	
Profit per unit($)	30	20	50	

The labor time per unit of model I is twice that of II and three times that of III. The entire labor force of the factory can produce the equivalent of 1500 units of model I. Market requirements specify the ratios 3:2:5 for the production of the three respective models. Formulate the problem as a linear program, and find the optimum solution.

4. The demand for ice cream during the three summer months (June, July, and August) at All-Flavors Parlor is estimated at 500, 600, and 400 20-gallon cartons, respectively. Two wholesalers, 1 and 2, supply All-Flavors with its ice cream. Although the flavors from the two suppliers are different, they are interchangeable. The maximum number of cartons either supplier can provide is 400 per month. Also, the prices the two suppliers charge change from one month to the next according to the following schedule:

	Price per carton in month		
	June	*July*	*August*
Supplier 1	$100	$110	$120
Supplier 2	$115	$108	$125

To take advantage of price fluctuation, All-Flavors can purchase more than is needed for a month and store the surplus to satisfy the demand in a later month. The cost of refrigerating an ice cream carton is $5 per month. It is realistic in the present situation to assume that the refrigeration cost is a function of the average number of cartons on hand during the month. Develop an optimum schedule for buying ice cream from the two suppliers.

5. The demand for an item over the next four quarters is 300, 400, 450, and 250 units, respectively. The price per unit starts at $20 in the first quarter and increases by $2 each quarter thereafter. The supplier can provide no more than 400 units in any one quarter. Although we can take advantage of lower prices in early quarters, a storage cost of $3.50 is incurred per unit per quarter. In addition, the maximum number of units that can be held over from one quarter to the next cannot exceed 100. Develop an optimum schedule for purchasing the item to meet the demand.

6. A company has contracted to produce two products, A and B, over the months of June, July, and August. The total production capacity (expressed in hours) varies monthly. The following table provides the basic data of the situation:

	June	July	August
Demand for *A* (units)	500	5000	750
Demand for *B* (units)	1000	1200	1200
Capacity (hours)	3000	3500	3000

The production rates in units per hour are 1.25 and 1 for products A and B, respectively. All demand must be met. However, demand for a later month may be filled from the production in an earlier one. For any carryover from one month to the next, holding costs of \$.90 and \$.75 per unit per month are charged for products A and B, respectively. The unit production costs for the two products are \$30 and \$28 for A and B, respectively. Determine the optimum production schedule for the two products.

*7. The manufacturing process of a product consists of two successive operations, I and II. The following table provides the pertinent data over the months of June, July, and August:

	June	July	August
Finished product demand (units)	500	450	600
Capacity of operation I (hr)	800	700	550
Capacity of operation II (hr)	1000	850	700

Producing a unit of the product takes .6 hour on operation I plus .8 hour on operation II. Overproduction of either the semifinished product (operation I) or the finished product (operation II) in any month is allowed for use in a later month. The corresponding holding costs are \$.20 and \$.40 per unit per month. The production cost varies by operation and by month. For operation 1, the unit production cost is \$10, \$12, and \$11 for June, July, and August. For operation 2, the corresponding unit production cost is \$15, \$18, and \$16. Determine the optimal production schedule for the two operations over the 3-month horizon.

8. Two products are manufactured sequentially on two machines. The time available on each machine is 8 hours per day and may be increased by up to 4 hours of overtime, if necessary, at an additional cost of \$100 per hour. The table below gives the production rate on the two machines as well as the price per unit of the two products. Determine the optimum production schedule and the recommended use of overtime, if any.

	Production rate (units/hr)	
	Product 1	*Product 2*
Machine 1	5	5
Machine 2	8	4
Price per unit (\$)	110	118

2.3.5 Blending and Refining

A number of LP applications deal with blending different input materials to produce products that meet certain specifications while minimizing cost or maximizing profit. The input materials could be ores, metal scraps, chemicals, or crude oils and the output products could be metal ingots, paints, or gasoline of various grades. This section presents a (simplified) model for oil refining. The process starts with distilling crude oil to produce intermediate gasoline stocks and then blending these stocks to produce final gasolines. The final products must satisfy certain quality specifications (such as octane rating). In addition, distillation capacities and demand limits can directly affect the level of production of the different grades of gasoline. One goal of the model is determine the optimal mix of final products that will maximize an appropriate profit function. In some cases, the goal may be to minimize a cost function.

Example 2.3-7 (Crude Oil Refining and Gasoline Blending)

Shale Oil, located on the island of Aruba, has a capacity of 1,500,000 bbl of crude oil per day. The final products from the refinery include three types of unleaded gasoline with different octane numbers (ON): regular with ON = 87, premium with ON = 89, and super with ON = 92. The refining process encompasses three stages: (1) a distillation tower that produces feedstock (ON = 82) at the rate of .2 bbl per bbl of crude oil, (2) a cracker unit that produces gasoline stock (ON = 98) by using a portion of the feedstock produced from the distillation tower at the rate of .5 bbl per bbl of feedstock, and (3) a blender unit that blends the gasoline stock from the cracker unit and the feedstock from the distillation tower. The company estimates the net profit per barrel of the three types of gasoline to be \$6.70, \$7.20, and \$8.10, respectively. The input capacity of the cracker unit is 200,000 barrels of feedstock a day. The demand limits for regular, premium, and super gasoline are 50,000, 30,000, and 40,000 barrels per day. Develop a model for determining the optimum production schedule for the refinery.

Mathematical Model: Figure 2.7 summarizes the elements of the model. The variables can be defined in terms of two input streams to the blender (feedstock and cracker gasoline) and the three final products. Let

$$x_{ij} = \text{bbl/day of input stream } i \text{ used to blend final product } j,\ i = 1, 2;\ j = 1, 2, 3$$

Using this definition, we have

$$\text{Daily production of regular gasoline} = x_{11} + x_{21} \text{ bbl/day}$$
$$\text{Daily production of premium gasoline} = x_{12} + x_{22} \text{ bbl/day}$$
$$\text{Daily production of super gasoline} = x_{13} + x_{23} \text{ bbl/day}$$

$$\begin{pmatrix}\text{Daily output}\\ \text{of blender unit}\end{pmatrix} = \begin{pmatrix}\text{Daily production}\\ \text{of regular gas}\end{pmatrix} + \begin{pmatrix}\text{Daily production}\\ \text{of premium gas}\end{pmatrix} + \begin{pmatrix}\text{Daily production}\\ \text{of super gas}\end{pmatrix}$$
$$= (x_{11} + x_{21}) + (x_{12} + x_{22}) + (x_{13} + x_{23})\text{bbl/day}$$

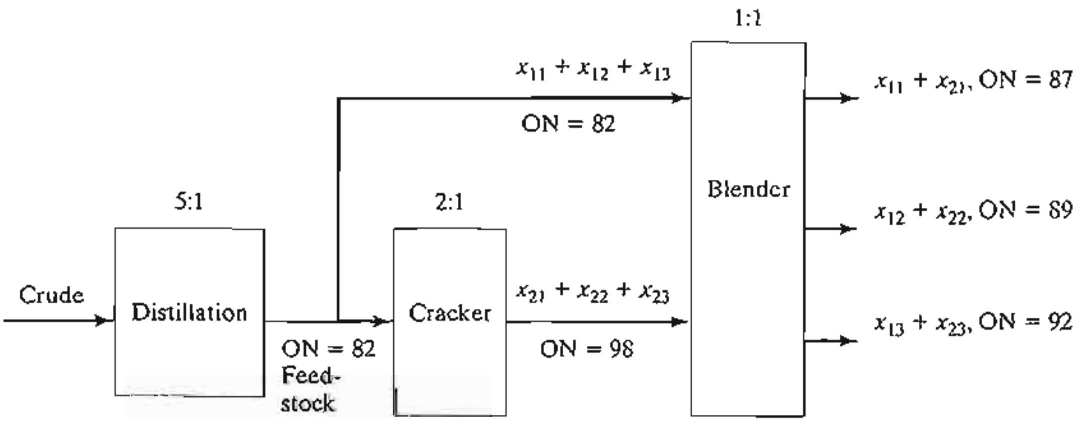


FIGURE 2.7

Product flow in the refinery problem

$$\begin{pmatrix}\text{Daily feedstock}\\ \text{to blender}\end{pmatrix} = x_{11} + x_{12} + x_{13} \text{ bbl/day}$$

$$\begin{pmatrix}\text{Daily cracker unit}\\ \text{feed to blender}\end{pmatrix} = x_{21} + x_{22} + x_{23} \text{ bbl/day}$$

$$\begin{pmatrix}\text{Daily feedstock}\\ \text{to cracker}\end{pmatrix} = 2(x_{21} + x_{22} + x_{23}) \text{bbl/day}$$

$$\begin{pmatrix}\text{Daily crude oil used}\\ \text{in the refinery}\end{pmatrix} = 5(x_{11} + x_{12} + x_{13}) + 10(x_{21} + x_{22} + x_{23}) \text{bbl/day}$$

The objective of the model is to maximize the total profit resulting from the sale of all three grades of gasoline. From the definitions given above, we get

$$\text{Maximize } z = 6.70(x_{11} + x_{21}) + 7.20(x_{12} + x_{22}) + 8.10(x_{13} + x_{23})$$

The constraints of the problem are developed as follows:

1. *Daily crude oil supply does not exceed 1,500,000 bbl/day:*

$$5(x_{11} + x_{12} + x_{13}) + 10(x_{21} + x_{22} + x_{23}) \leq 1{,}500{,}000$$

2. *Cracker unit input capacity does not exceed 200,000 bbl/day:*

$$2(x_{21} + x_{22} + x_{23}) \leq 200{,}000$$

3. *Daily demand for regular does not exceed 50,000 bbl:*

$$x_{11} + x_{21} \leq 50{,}000$$

4. *Daily demand for premium does not exceed 30,000:*

$$x_{12} + x_{22} \leq 30{,}000$$

5. *Daily demand for super does not exceed 40,000 bbl:*

$$x_{13} + x_{23} \leq 40{,}000$$

6. *Octane number (ON) for regular is at least 87:*

   The octane number of a gasoline product is the weighted average of the octane numbers of the input streams used in the blending process and can be computed as

$$\begin{pmatrix}\text{Average ON of}\\ \text{regular gasoline}\end{pmatrix} = \frac{\text{Feedstock ON} \times \text{feedstock bbl/day} + \text{Cracker unit ON} \times \text{Cracker unit bbl/day}}{\text{Total bbl/day of regular gasoline}}$$

$$= \frac{82x_{11} + 98x_{21}}{x_{11} + x_{21}}$$

   Thus, octane number constraint for regular gasoline becomes

$$\frac{82x_{11} + 98x_{21}}{x_{11} + x_{21}} \geq 87$$

The constraint is linearized as

$$82x_{11} + 98x_{21} \geq 87(x_{11} + x_{21})$$

7. *Octane number (ON) for premium is at least 89:*

$$\frac{82x_{12} + 98x_{22}}{x_{12} + x_{22}} \geq 89$$

which is linearized as

$$82x_{12} + 98x_{22} \geq 89(x_{12} + x_{22})$$

8. *Octane number (ON) for super is at least 92:*

$$\frac{82x_{13} + 98x_{23}}{x_{13} + x_{23}} \geq 92$$

or

$$82x_{13} + 98x_{23} \geq 92(x_{13} + x_{23})$$

The complete model is thus summarized as

$$\text{Maximize } z = 6.70(x_{11} + x_{21}) + 7.20(x_{12} + x_{22}) + 8.10(x_{13} + x_{23})$$

subject to

$$\begin{aligned}
5(x_{11} + x_{12} + x_{13}) + 10(x_{21} + x_{22} + x_{23}) &\leq 1{,}500{,}000 \\
2(x_{21} + x_{22} + x_{23}) &\leq 200{,}000 \\
x_{11} + x_{21} &\leq 50{,}000 \\
x_{12} + x_{22} &\leq 30{,}000 \\
x_{13} + x_{23} &\leq 40{,}000 \\
82x_{11} + 98x_{21} &\geq 87(x_{11} + x_{21}) \\
82x_{12} + 98x_{22} &\geq 89(x_{12} + x_{22}) \\
82x_{13} + 98x_{23} &\geq 92(x_{13} + x_{23})
\end{aligned}$$

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0$$

The last three constraints can be simplified to produce a constant right-hand side.

Solution:

The optimum solution (using file amplEx2.3-7.txt) is $z = 1{,}482{,}000$, $x_{11} = 20{,}625$, $x_{21} = 9375$, $x_{12} = 16{,}875$, $x_{22} = 13{,}125$, $x_{13} = 15{,}000$, $x_{23} = 25{,}000$. This translates to

Daily profit = \$1,482,000

Daily amount of regular gasoline = $x_{11} + x_{21} = 20{,}625 + 9375 = 30{,}000$ bbl/day

Daily amount of premium gasoline = $x_{12} + x_{22} = 16{,}875 + 13{,}125 = 30{,}000$ bbl/day

Daily amount of regular gasoline = $x_{13} + x_{23} = 15{,}000 + 25{,}000 = 40{,}000$ bbl/day

The solution shows that regular gasoline production is 20,000 bbl/day short of satisfying the maximum demand. The demand for the remaining two grades is satisfied.

PROBLEM SET 2.3E

1. Hi-V produces three types of canned juice drinks, *A*, *B*, and *C*, using fresh strawberries, grapes, and apples. The daily supply is limited to 200 tons of strawberries, 100 tons of grapes, and 150 tons of apples. The cost per ton of strawberries, grapes, and apples is $200, $100, and $90, respectively. Each ton makes 1500 lb of strawberry juice, 1200 lb of grape juice, and 1000 lb of apple juice. Drink *A* is a 1:1 mix of strawberry and apple juice. Drink *B* is 1:1:2 mix of strawberry, grape, and apple juice. Drink *C* is a 2:3 mix of grape and apple juice. All drinks are canned in 16-oz (1 lb) cans. The price per can is $1.15, $1.25, and $1.20 for drinks *A*, *B*, and *C*. Determine the optimal production mix of the three drinks.

*2. A hardware store packages handyman bags of screws, bolts, nuts, and washers. Screws come in 100-lb boxes and cost $110 each, bolts come in 100-lb boxes and cost $150 each, nuts come in 80-lb boxes and cost $70 each, and washers come in 30-lb boxes and cost $20 each. The handyman package weighs at least 1 lb and must include, by weight, at least 10% screws and 25% bolts, and at most 15% nuts and 10% washers. To balance the package, the number of bolts cannot exceed the number of nuts or the number of washers. A bolt weighs 10 times as much as a nut and 50 times as much as a washer. Determine the optimal mix of the package.

3. All-Natural Coop makes three breakfast cereals, *A*, *B*, and *C*, from four ingredients: rolled oats, raisins, shredded coconuts, and slivered almonds. The daily availabilities of the ingredients are 5 tons, 2 tons, 1 ton, and 1 ton, respectively. The corresponding costs per ton are $100, $120, $110, and $200. Cereal *A* is a 50:5:2 mix of oats, raisins, and almond. Cereal *B* is a 60:2:3 mix of oats, coconut, and almond. Cereal *C* is a 60:3:4:2 mix of oats, raisins, coconut, and almond. The cereals are produced in jumbo 5-lb sizes. All-Natural sells *A*, *B*, and *C* at $2, $2.50, and $3.00 per box, respectively. The minimum daily demand for cereals *A*, *B*, and *C* is 500, 600, and 500 boxes. Determine the optimal production mix of the cereals and the associated amounts of ingredients.

4. A refinery manufactures two grades of jet fuel, *F1* and *F2*, by blending four types of gasoline, *A*, *B*, *C*, and *D*. Fuel *F1* uses gasolines *A*, *B*, *C*, and *D* in the ratio 1:1:2:4, and fuel *F2* uses the ratio 2:2:1:3. The supply limits for *A*, *B*, *C*, and *D* are 1000, 1200, 900, and 1500 bbl/day, respectively. The costs per bbl for gasolines *A*, *B*, *C*, and *D* are $120, $90, $100, and $150, respectively. Fuels *F1* and *F2* sell for $200 and $250 per bbl. The minimum demand for *F1* and *F2* is 200 and 400 bbl/day. Determine the optimal production mix for *F1* and *F2*.

5. An oil company distills two types of crude oil, *A* and *B*, to produce regular and premium gasoline and jet fuel. There are limits on the daily availability of crude oil and the minimum demand for the final products. If the production is not sufficient to cover demand, the shortage must be made up from outside sources at a penalty. Surplus production will not be sold immediately and will incur storage cost. The following table provides the data of the situation:

	Fraction yield per bbl				
Crude	*Regular*	*Premium*	*Jet*	Price/bbl ($)	bbl/day
Crude A	.20	.1	.25	30	2500
Crude B	.25	.3	.10	40	3000
Demand (bbl/day)	500	700	400		
Revenue ($/bbl)	50	70	120		
Storage cost for surplus production ($/bbl)	2	3	4		
Penalty for unfilled demand ($/bbl)	10	15	20		

Determine the optimal product mix for the refinery.

6. In the refinery situation of Problem 5, suppose that the distillation unit actually produces the intermediate products naphtha and light oil. One bbl of crude A produces .35 bbl of naphtha and .6 bbl of light oil, and one bbl of crude B produces .45 bbl of naphtha and .5 bbl of light oil. Naphtha and light oil are blended to produce the three final gasoline products: One bbl of regular gasoline has a blend ratio of 2:1 (naphtha to light oil), one bbl of premium gasoline has a blend ratio of ratio of 1:1, and one bbl of jet fuel has a blend ratio of 1:2. Determine the optimal production mix.
7. Hawaii Sugar Company produces brown sugar, processed (white) sugar, powdered sugar, and molasses from sugar cane syrup. The company purchases 4000 tons of syrup weekly and is contracted to deliver at least 25 tons weekly of each type of sugar. The production process starts by manufacturing brown sugar and molasses from the syrup. A ton of syrup produces .3 ton of brown sugar and .1 ton of molasses. White sugar is produced by processing brown sugar. It takes 1 ton of brown sugar to produce .8 ton of white sugar. Powdered sugar is produced from white sugar through a special grinding process that has a 95% conversion efficiency (1 ton of white sugar produces .95 ton of powdered sugar). The profits per ton for brown sugar, white sugar, powdered sugar, and molasses are \$150, \$200, \$230, and \$35, respectively. Formulate the problem as a linear program, and determine the weekly production schedule.
8. Shale Oil refinery blends two petroleum stocks, A and B, to produce two high-octane gasoline products, I and II. Stocks A and B are produced at the maximum rates of 450 and 700 bbl/hour, respectively. The corresponding octane numbers are 98 and 89, and the vapor pressures are 10 and 8 lb/in^2. Gasoline I and gasoline II must have octane numbers of at least 91 and 93, respectively. The vapor pressure associated with both products should not exceed 12 lb/in^2. The profits per bbl of I and II are \$7 and \$10, respectively. Determine the optimum production rate for I and II and their blend ratios from stocks A and B. (*Hint:* Vapor pressure, like the octane number, is the weighted average of the vapor pressures of the blended stocks.)
9. A foundry smelts steel, aluminum, and cast iron scraps to produce two types of metal ingots, I and II, with specific limits on the aluminum, graphite and silicon contents. Aluminum and silicon briquettes may be used in the smelting process to meet the desired specifications. The following tables set the specifications of the problem: 2.3

	Contents (%)				
Input item	*Aluminum*	*Graphite*	*Silicon*	Cost/ton (\$)	Available tons/day
Steel scrap	10	5	4	100	1000
Aluminum scrap	95	1	2	150	500
Cast iron scrap	0	15	8	75	2500
Aluminum briquette	100	0	0	900	Any amount
Silicon briquette	0	0	100	380	Any amount

	Ingot I		Ingot II	
Ingredient	*Minimum*	*Maximum*	*Minimum*	*Maximum*
Aluminum	8.1%	10.8%	6.2%	8.9%
Graphite	1.5%	3.0%	4.1%	∞
Silicon	2.5%	∞	2.8%	4.1%
Demand (tons/day)	130		250	

Determine the optimal input mix the foundry should smelt.

10. Two alloys, A and B, are made from four metals, I, II, III, and IV, according to the following specifications:

Alloy	Specifications	Selling price ($)
A	At most 80% of I At most 30% of II At least 50% of IV	200
B	Between 40% and 60% of II At least 30% of III At most 70% of IV	300

The four metals, in turn, are extracted from three ores according to the following data:

		Constituents (%)					
Ore	Maximum quantity (tons)	*I*	*II*	*III*	*IV*	*Others*	Price/ton ($)
1	1000	20	10	30	30	10	30
2	2000	10	20	30	30	10	40
3	3000	5	5	70	20	0	50

How much of each type of alloy should be produced? (*Hint:* Let x_{kj} be tons of ore i allocated to alloy k, and define w_k as tons of alloy k produced.)

2.3.6 Manpower Planning

Fluctuations in a labor force to meet variable demand over time can be achieved through the process of hiring and firing, as demonstrated in Example 2.3-6. There are situations in which the effect of fluctuations in demand can be "absorbed" by adjusting the start and end times of a work shift. For example, instead of following the traditional three 8-hour-shift start times at 8:00 A.M., 3:00 P.M., and 11:00 P.M., we can use overlapping 8-hour shifts in which the start time of each is made in response to increase or decrease in demand.

The idea of redefining the start of a shift to accommodate fluctuation in demand can be extended to other operating environments as well. Example 2.3-8 deals with the determination of the minimum number of buses needed to meet rush-hour and off-hour transportation needs.

Real-Life Application—Telephone Sales Manpower Planning at Qantas Airways

Australian airline Qantas operates its main reservation offices from 7:00 till 22:00 using 6 shifts that start at different times of the day. Qantas used linear programming (with imbedded queuing analysis) to staff its main telephone sales reservation office efficiently while providing convenient service to its customers. The study, carried out in the late 1970s, resulted in annual savings of over 200,000 Australian dollars per year. The study is detailed in Case 15, Chapter 24 on the CD.

Example 2.3-8 (Bus Scheduling)

Progress City is studying the feasibility of introducing a mass-transit bus system that will alleviate the smog problem by reducing in-city driving. The study seeks the minimum number of buses that can handle the transportation needs. After gathering necessary information, the city engineer noticed that the minimum number of buses needed fluctuated with the time of the day and that the required number of buses could be approximated by constant values over successive 4-hour intervals. Figure 2.8 summarizes the engineer's findings. To carry out the required daily maintenance, each bus can operate 8 successive hours a day only.

Mathematical Model: Determine the number of operating buses in each shift (variables) that will meet the minimum demand (constraints) while minimizing the total number of buses in operation (objective).

You may already have noticed that the definition of the variables is ambiguous. We know that each bus will run for 8 consecutive hours, but we do not know when a shift should start. If we follow a normal three-shift schedule (8:01 A.M.-4:00 P.M., 4:01 P.M.-12:00 midnight, and 12:01 A.M.-8:00 A.M.) and assume that x_1, x_2, and x_3 are the number of buses starting in the first, second, and third shifts, we can see from Figure 2.8 that $x_1 \geq 10$, $x_2 \geq 12$, and $x_3 \geq 8$. The corresponding minimum number of daily buses is $x_1 + x_2 + x_3 = 10 + 12 + 8 = 30$.

The given solution is acceptable only if the shifts *must* coincide with the normal three-shift schedule. It may be advantageous, however, to allow the optimization process to choose the "best" starting time for a shift. A reasonable way to accomplish this is to allow a shift to start every 4 hours. The bottom of Figure 2.8 illustrates this idea where overlapping 8-hour shifts

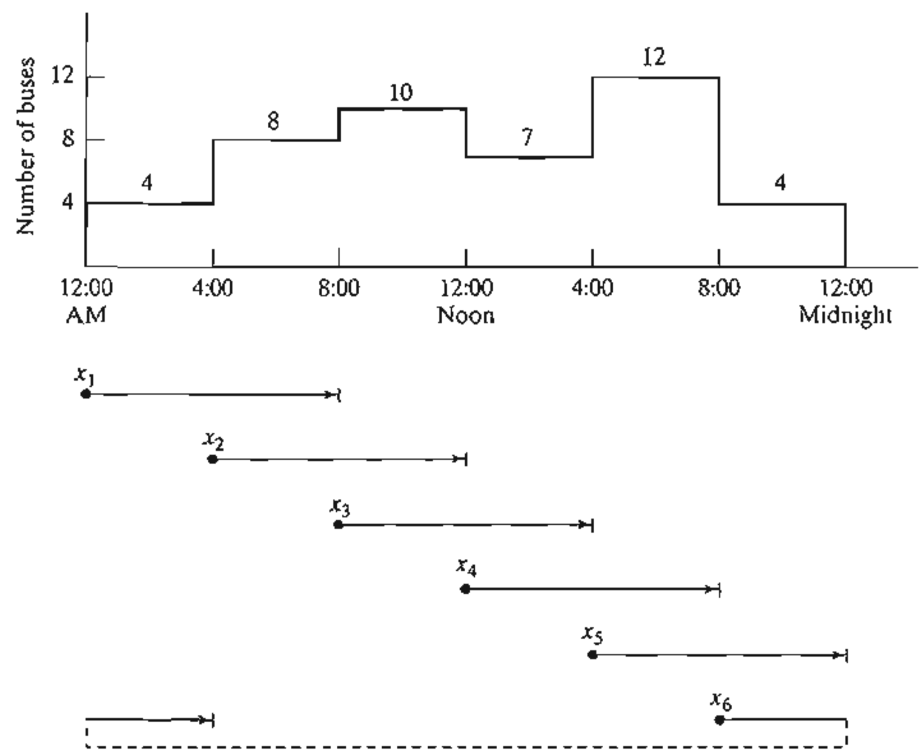


FIGURE 2.8

Number of buses as a function of the time of the day may start at 12:01 A.M., 4:01 A.M., 8:01 A.M., 12:01 P.M., 4:01 P.M., and 8:01 P.M. Thus, the variables may be defined as

x_1 = number of buses starting at 12: 01 A.M.

x_2 = number of buses starting at 4:01 A.M.

x_3 = number of buses starting at 8:01 A.M.

x_4 = number of buses starting at 12:01 P.M.

x_5 = number of buses starting at 4:01 P.M.

x_6 = number of buses starting at 8:01 P.M.

We can see from Figure 2.8 that because of the overlapping of the shifts, the number of buses for the successive 4-hour periods is given as

Time period	Number of buses in operation
12:01 A.M. – 4:00 A.M.	$x_1 + x_6$
4:01 A.M. – 8:00 A.M.	$x_1 + x_2$
8:01 A.M. – 12:00 noon	$x_2 + x_3$
12:01 P.M. – 4:00 P.M.	$x_3 + x_4$
4:01 P.M. – 8:00 P.M.	$x_4 + x_5$
8:01 A.M. – 12:00 A.M.	$x_5 + x_6$

The complete model is thus written as

$$\text{Minimize } z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

subject to

$$\begin{aligned}
x_1 \qquad\qquad\qquad\qquad + x_6 &\geq 4 \text{ (12:01 A.M.-4:00 A.M.)}\\
x_1 + x_2 &\geq 8 \text{ (4:01 A.M.-8:00 A.M.)}\\
x_2 + x_3 &\geq 10 \text{ (8:01 A.M.-12:00 noon)}\\
x_3 + x_4 &\geq 7 \text{ (12:01 P.M.-4:00 P.M.)}\\
x_4 + x_5 &\geq 12 \text{ (4:01 P.M.-8:00 P.M.)}\\
x_5 + x_6 &\geq 4 \text{ (8:01 P.M.-12:00 P.M.)}\\
x_j \geq 0, j &= 1, 2, \ldots, 6
\end{aligned}$$

Solution:

The optimal solution calls for using 26 buses to satisfy the demand with $x_1 = 4$ buses to start at 12:01 A.M., $x_2 = 10$ at 4:01 A.M., $x_4 = 8$ at 12:01 P.M., and $x_5 = 4$ at 4:01 P.M.

PROBLEM SET 2.3F

*1. In the bus scheduling example suppose that buses can run either 8- or 12-hour shifts. If a bus runs for 12 hours, the driver must be paid for the extra hours at 150% of the regular hourly pay. Do you recommend the use of 12-hour shifts?

2. A hospital employs volunteers to staff the reception desk between 8:00 A.M. and 10:00 P.M. Each volunteer works three consecutive hours except for those starting at 8:00 P.M. who work for two hours only. The minimum need for volunteers is approximated by a step function over 2-hour intervals starting at 8:00 A.M. as 4, 6, 8, 6, 4, 6, 8. Because most volunteers are retired individuals, they are willing to offer their services at any hour of the day (8:00 A.M. to 10:00 P.M.). However, because of the large number of charities competing for their service, the number needed must be kept as low as possible. Determine an optimal schedule for the start time of the volunteers
3. In Problem 2, suppose that no volunteers will start at noon or 6:00 P.M. to allow for lunch and dinner. Determine the optimal schedule.
4. In an LTL (less-than-truckload) trucking company, terminal docks include *casual* workers who are hired temporarily to account for peak loads. At the Omaha, Nebraska, dock, the minimum demand for casual workers during the seven days of the week (starting on Monday) is 20, 14, 10, 15, 18, 10, 12 workers. Each worker is contracted to work five consecutive days. Determine an optimal weekly hiring practice of casual workers for the company.

*5. On most university campuses students are contracted by academic departments to do errands, such as answering the phone and typing. The need for such service fluctuates during work hours (8:00 A.M. to 5:00 P.M.). In the IE department, the minimum number of students needed is 2 between 8:00 A.M. and 10:00 A.M., 3 between 10:01 A.M. and 11:00 A.M., 4 between 11:01 A.M. and 1:00 P.M., and 3 between 1:01 P.M. and 5:00 P.M. Each student is allotted 3 consecutive hours (except for those starting at 3:01, who work for 2 hours and those who start at 4:01, who work for one hour). Because of their flexible schedule, students can usually report to work at any hour during the work day, except that no student wants to start working at lunch time (12:00 noon). Determine the minimum number of students the IE department should employ and specify the time of the day at which they should report to work.

6. A large department store operates 7 days a week. The manager estimates that the minimum number of salespersons required to provide prompt service is 12 for Monday, 18 for Tuesday, 20 for Wednesday, 28 for Thursday, 32 for Friday, and 40 for each of Saturday and Sunday. Each salesperson works 5 days a week, with the two consecutive off-days staggered throughout the week. For example, if 10 salespersons start on Monday, two can take their off-days on Tuesday and Wednesday, five on Wednesday and Thursday, and three on Saturday and Sunday. How many salespersons should be contracted and how should their off-days be allocated?

2.3.7 Additional Applications

The preceding sections have demonstrated the application of LP to six representative areas. The fact is that LP enjoys diverse applications in an enormous number of areas. The problems at the end of this section demonstrate some of these areas, ranging from agriculture to military applications. This section also presents an interesting application that deals with cutting standard stocks of paper rolls to sizes specified by customers.

Example 2.3-9 (Trim Loss or Stock Slitting)

The Pacific Paper Company produces paper rolls with a standard width of 20 feet each. Special customer orders with different widths are produced by slitting the standard rolls. Typical orders (which may vary daily) are summarized in the following table:

Order	Desired width (ft)	Desired number of rolls
1	5	150
2	7	200
3	9	300

In practice, an order is filled by setting the knives to the desired widths. Usually, there are a number of ways in which a standard roll may be slit to fill a given order. Figure 2.9 shows three feasible knife settings for the 20-foot roll. Although there are other feasible settings, we limit the discussion for the moment to settings 1, 2, and 3 in Figure 2.9. We can combine the given settings in a number of ways to fill orders for widths 5, 7, and 9 feet. The following are examples of feasible combinations:

1. Slit 300 (standard) rolls using setting 1 and 75 rolls using setting 2.
2. Slit 200 rolls using setting 1 and 100 rolls using setting 3.

Which combination is better? We can answer this question by considering the "waste" each combination generates. In Figure 2.9 the shaded portion represents surplus rolls not wide enough to fill the required orders. These surplus rolls are referred to as *trim loss*. We can evaluate the "goodness" of each combination by computing its trim loss. However, because the surplus rolls may have different widths, we should base the evaluation on the trim loss *area* rather than on the *number* of surplus rolls. Assuming that the standard roll is of length L feet, we can compute the trim-loss area as follows:

$$\text{Combination 1: } 300\,(4 \times L) + 75\,(3 \times L) = 1425L \text{ ft}^2$$

$$\text{Combination 2: } 200\,(4 \times L) + 100\,(1 \times L) = 900L \text{ ft}^2$$

These areas account only for the shaded portions in Figure 2.9. Any surplus production of the 5-, 7- and 9-foot rolls must be considered also in the computation of the trim-loss area. In

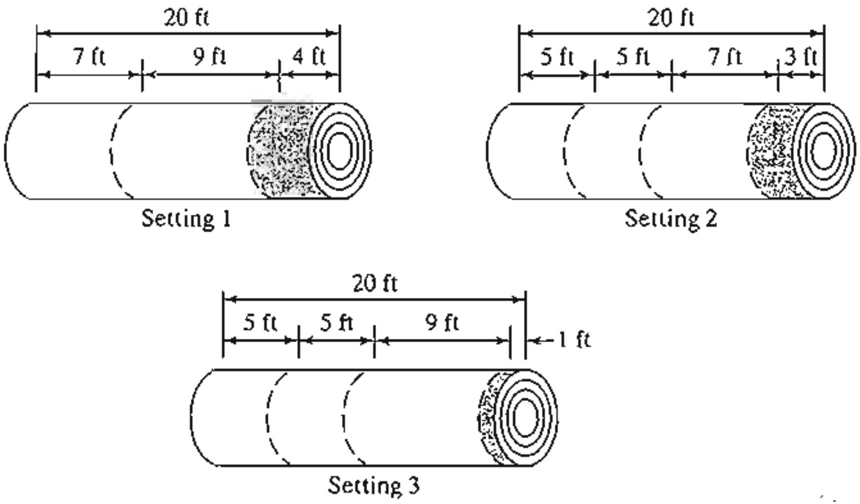


FIGURE 2.9

Trim loss (shaded) for knife settings 1, 2, and 3 combination 1, setting 1 produces a surplus of $300 - 200 = 100$ extra 7-foot rolls and setting 2 produces 75 extra 7-foot rolls. Thus the additional waste area is $175\ (7 \times L) = 1225L\ \text{ft}^2$. Combination 2 does not produce surplus rolls of the 7- and 9-foot rolls but setting 3 does produce $200 - 150 = 50$ extra 5-foot rolls, with an added waste area of $50\ (5 \times L) = 250L\ \text{ft}^2$. As a result we have

$$\text{Total trim-loss area for combination } 1 = 1425L + 1225L = 2650L\ \text{ft}^2$$

$$\text{Total trim-loss area for combination } 2 = 900L + 250L = 1150L\ \text{ft}^2$$

Combination 2 is better, because it yields a smaller trim-loss area.

Mathematical Model: The problem can be summarized verbally as determining the *knife-setting combinations* (variables) that will *fill the required orders* (constraints) with the *least trim-loss area* (objective).

The definition of the variables as given must be translated in a way that the mill operator can use. Specifically, the variables are defined as *the number of standard rolls to be slit according to a given knife setting*. This definition requires identifying all possible knife settings as summarized in the following table (settings 1, 2, and 3 are given in Figure 2.9). You should convince yourself that settings 4, 5, and 6 are valid and that no "promising" settings have been excluded. Remember that a promising setting cannot yield a trim-loss roll of width 5 feet or larger.

	Knife setting						
Required width (ft)	*1*	*2*	*3*	*4*	*5*	*6*	Minimum number of rolls
5	0	2	2	4	1	0	150
7	1	1	0	0	2	0	200
9	1	0	1	0	0	2	300
Trim loss per foot of length	4	3	1	0	1	2	

To express the model mathematically, we define the variables as

$$x_j = \text{number of standard rolls to be slit according to setting } j,\ j = 1, 2, \ldots, 6$$

The constraints of the model deal directly with satisfying the demand for rolls.

$$\begin{aligned}
\text{Number of 5-ft rolls produced} &= \qquad\quad 2x_2 + 2x_3 + 4x_4 + x_5 \geq 150 \\
\text{Number of 7-ft rolls produced} &= x_1 + x_2 \qquad\qquad\quad + 2x_5 \geq 200 \\
\text{Number of 9-ft rolls produced} &= x_1 \quad x_2 + x_3 \qquad\quad + 2x_5 + 2x_6 \geq 300
\end{aligned}$$

To construct the objective function, we observe that the total trim loss area is the difference between the total area of the standard rolls used and the total area representing all the orders. Thus

$$\text{Total area of standard rolls} = 20L(x_1 + x_2 + x_3 + x_4 + x_5 + x_6)$$

$$\text{Total area of orders} = L(150 \times 5 + 200 \times 7 + 300 \times 9) = 4850L$$

The objective function then becomes

$$\text{Minimize } z = 20L(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) - 4850L$$

Because the length L of the standard roll is a constant, the objective function equivalently reduces to

$$\text{Minimize } z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

The model may thus be written as

$$\text{Minimize } z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

subject to

$$\begin{aligned}
2x_2 + 2x_3 + 4x_4 + x_5 \quad &\geq 150 \text{ (5-ft rolls)}\\
x_1 + x_2 + 2x_5 \quad &\geq 200 \text{ (7-ft rolls)}\\
x_1 + x_3 + 2x_6 \quad &\geq 300 \text{ (9-ft rolls)}\\
x_j \geq 0, j = 1, 2, \ldots, 6 &
\end{aligned}$$

Solution:

The optimum solution calls for cutting 12.5 standard rolls according to setting 4, 100 according to setting 5, and 150 according to setting 6. The solution is not implementable because x_4 is noninteger. We can either use an integer algorithm to solve the problem (see Chapter 9) or round x_4 conservatively to 13 rolls.

Remarks. The trim-loss model as presented here assumes that all the feasible knife settings can be determined in advance. This task may be difficult for large problems, and viable feasible combinations may be missed. The problem can be remedied by using an LP model with imbedded integer programs designed to generate promising knife settings on demand until the optimum solution is found. This algorithm, sometimes referred to as **column generation**, is detailed in Comprehensive Problem 7-3, Appendix E on the CD. The method is rooted in the use of (reasonably advanced) linear programming *theory*, and may serve to refute the argument that, in practice, it is unnecessary to learn LP theory.

PROBLEM SET 2.3G

*1. Consider the trim-loss model of Example 2.3-9.

(a) If we slit 200 rolls using setting 1 and 100 rolls using setting 3, compute the associated trim-loss area.

(b) Suppose that the only available standard roll is 15 feet wide. Generate all possible knife settings for producing 5-, 7-, and 9-foot rolls, and compute the associated trim loss per foot length.

(c) In the original model, if the demand for 7-foot rolls is decreased by 80, what is the minimum number of standard 20-foot rolls that will be needed to fill the demand for of all three types of rolls?

(d) In the original model, if the demand for 9-foot rolls is changed to 400, how many additional standard 20-foot rolls will be needed to satisfy the new demand?

2. *Shelf Space Allocation.* A grocery store must decide on the shelf space to be allocated to each of five types of breakfast cereals. The maximum daily demand is 100, 85, 140, 80, and 90 boxes, respectively. The shelf space in square inches for the respective boxes is 16, 24, 18, 22, and 20. The total available shelf space is 5000 in^2. The profit per unit is \$1.10, \$1.30, \$1.08, \$1.25, and \$1.20, respectively. Determine the optimal space allocation for the five cereals.
3. *Voting on Issues.* In a particular county in the State of Arkansas, four election issues are on the ballot: Build new highways, increase gun control, increase farm subsidies, and increase gasoline tax. The county includes 100,000 urban voters, 250,000 suburban voters, and 50,000 rural voters, all with varying degrees of support for and opposition to election issues. For example, rural voters are opposed to gun control and gas tax and in favor of road building and farm subsidies. The county is planning a TV advertising campaign with a budget of \$100,000 at the cost of \$1500 per ad. The following table summarizes the impact of a single ad in terms of the number of pro and con votes as a function of the different issues:

	Expected number of pro (+) and con (−) votes per ad		
Issue	*Urban*	*Suburban*	*Rural*
New highways	−30,000	+60,000	+30,000
Gun control	+80,000	+30,000	−45,000
Smog control	+40,000	+10,000	0
Gas tax	+90,000	0	−25,000

An issue will be adopted if it garners at least 51% of the votes. Which issues will be approved by voters, and how many ads should be allocated to these issues?

4. *Assembly-Line Balancing.* A product is assembled from three different parts. The parts are manufactured by two departments at different production rates as given in the following table:

		Production rate (units/hr)		
Department	Capacity (hr/wk)	*Part 1*	*Part 2*	*Part 3*
1	100	8	5	10
2	80	6	12	4

Determine the maximum number of final assembly units that can be produced weekly. (*Hint:* Assembly units = min {units of part 1, units of part 2, units of part 3}. Maximize $z = \min\{x_1, x_2\}$ is equivalent to max z subject to $z \le x_1$ and $z \le x_2$.)

5. *Pollution Control.* Three types of coal, C1, C2, and C3, are pulverized and mixed together to produce 50 tons per hour needed to power a plant for generating electricity. The burning of coal emits sulfur oxide (in parts per million) which must meet the Environmental Protection Agency (EPA) specifications of at most 2000 parts per million. The following table summarizes the data of the situation:

	C1	C2	C3
Sulfur (parts per million)	2500	1500	1600
Pulverizer capacity (ton/hr)	30	30	30
Cost per ton	\$30	\$35	\$33

Determine the optimal mix of the coals.

*6. *Traffic Light Control.* (Stark and Nicholes, 1972) Automobile traffic from three highways, H1, H2, and H3, must stop and wait for a green light before exiting to a toll road. The tolls are \$3, \$4, and \$5 for cars exiting from H1, H2, and H3, respectively. The flow rates from H1, H2, and H3 are 500, 600, and 400 cars per hour. The traffic light cycle may not exceed 2.2 minutes, and the green light on any highway must be at least 25 seconds. The yellow light is on for 10 seconds. The toll gate can handle a maximum of 510 cars per hour. Assuming that no cars move on yellow, determine the optimal green time interval for the three highways that will maximize toll gate revenue per traffic cycle.

7. *Fitting a Straight Line into Empirical Data (Regression).* In a 10-week typing class for beginners, the average speed per student (in words per minute) as a function of the number of weeks in class is given in the following table.

Week, x	1	2	3	4	5	6	7	8	9	10
Words per minute, y	5	9	15	19	21	24	26	30	31	35

Determine the coefficients a and b in the straight-line relationship, $\hat{y} = ax + b$, that best fit the given data. (*Hint:* Minimize the sum of the *absolute value* of the deviations between theoretical $\hat{y}$ and empirical y. Min $|x|$ is equivalent to min z subject to $z \leq x$ and $z \geq -x$.)

8. *Leveling the Terrain for a New Highway.* (Stark and Nicholes, 1972) The Arkansas Highway Department is planning a new 10-mile highway on uneven terrain as shown by the profile in Figure 2.10. The width of the construction terrain is approximately 50 yards. To simplify the situation, the terrain profile can be replaced by a step function as shown in the figure. Using heavy machinery, earth removed from high terrain is hauled to fill low areas. There are also two burrow pits, I and II, located at the ends of the 10-mile stretch from which additional earth can be hauled, if needed. Pit I has a capacity of 20,000 cubic yards and pit II a capacity of 15,000 cubic yards. The costs of removing earth from pits I and II are, respectively, \$1.50 and \$1.90 per cubic yard. The transportation cost per cubic yard per mile is \$.15 and the cost of using heavy machinery to load hauling trucks is \$.20 per cubic yard. This means that a

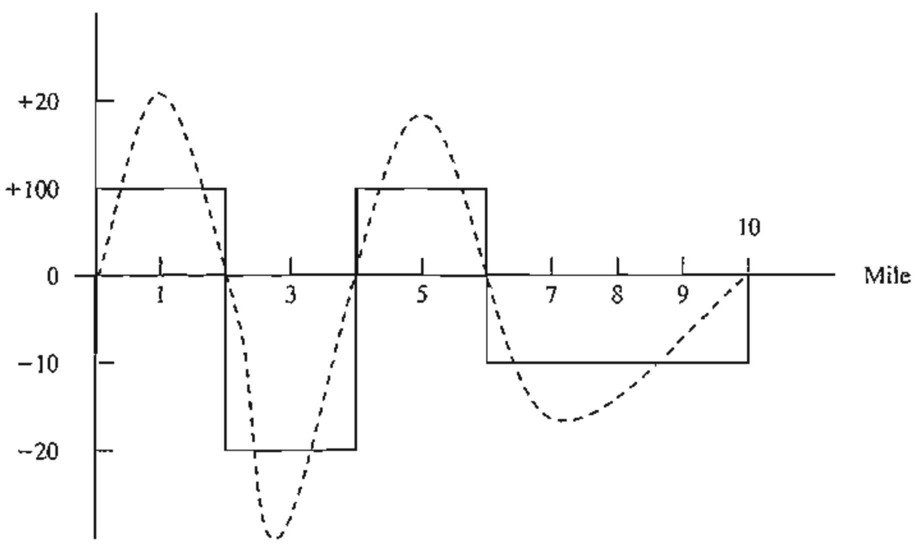


FIGURE 2.10
Terrain profile for Problem 8

cubic yard from pit I hauled one mile will cost a total of $(1.5 + .20) + 1 \times .15 = \$1.85$ and a cubic yard hauled one mile from a hill to a fill area will cost $.20 + 1 \times .15 = \$.35$. Develop a minimum cost plan for leveling the 10-mile stretch.

9. *Military Planning.* (Shepard and Associates, 1988) The Red Army (R) is trying to invade the territory defended by the Blue Army (B). Blue has three defense lines and 200 regular combat units and can draw also on a reserve pool of 200 units. Red plans to attack on two fronts, north and south, and Blue has set up three east-west defense lines, I, II, and III. The purpose of defense lines I and II is to delay the Red Army attack by at least 4 days in each line and to maximize the total duration of the battle. The advance time of the Red Army is estimated by the following empirical formula:

$$\text{Battle duration in days} = a + b\left(\frac{\text{Blue units}}{\text{Red units}}\right)$$

The constants a and b are a function of the defense line and the north/south front as the following table shows:

	a			b		
	I	*II*	*III*	*I*	*II*	*III*
North front	.5	.75	.55	8.8	7.9	10.2
South front	1.1	1.3	1.5	10.5	8.1	9.2

The Blue Army reserve units can be used in defense lines II and II only. The allocation of units by the Red Army to the three defense lines is given in the following table.

	Number of Red Army attack units		
	Defense Line I	*Defense Line II*	*Defense Line III*
North front	30	60	20
South front	30	40	20

How should Blue allocate its resources among the three defense lines and the north/south fronts?

10. *Water Quality Management.* (Stark and Nicholes, 1972) Four cities discharge waste water into the same stream. City 1 is upstream, followed downstream by city 2, then city 3, then city 4. Measured alongside the stream, the cities are approximately 15 miles apart. A measure of the amount of pollutants in waste water is the BOD (biochemical oxygen demand), which is the weight of oxygen required to stabilize the waste constituent in water. A higher BOD indicates worse water quality. The Environmental Protection Agency (EPA) sets a maximum allowable BOD loading, expressed in lb BOD per gallon. The removal of pollutants from waste water takes place in two forms: (1) natural decomposition activity stimulated by the oxygen in the air, and (2) treatment plants at the points of discharge before the waste reaches the stream. The objective is to determine the most economical efficiency of each of the four plants that will reduce BOD to acceptable levels. The maximum possible plant efficiency is 99%.

To demonstrate the computations involved in the process, consider the following definitions for plant 1:

Q_1 = Stream flow (gal/hour) on the 15-mile reach 1-2 leading to city 2

p_1 = BOD discharge rate (in lb/hr)

x_1 = efficiency of plant 1 ($\leq .99$)
b_1 = maximum allowable BOD loading in reach 1-2 (in lb BOD/gal)

To satisfy the BOD loading requirement in reach 1-2, we must have

$$p_1(1 - x_1) \leq b_1 Q_1$$

In a similar manner, the BOD loading constraint for reach 2-3 takes the form

$$(1 - r_{12})\begin{pmatrix}\text{BOD discharge}\\ \text{rate in reach 1-2}\end{pmatrix} + \begin{pmatrix}\text{BOD discharge}\\ \text{rate in reach 2-3}\end{pmatrix} \leq b_2 Q_2$$

or

$$(1 - r_{12})p_1(1 - x_1) + p_2(1 - x_2) \leq b_2 Q_2$$

The coefficient r_{12} (<1) represents the fraction of waste removed in reach 1-2 by decomposition. For reach 2-3, the constraint is

$$(1 - r_{23})[(1 - r_{12})p_1(1 - x_1) + p_2(1 - x_2)] + p_3(1 - x_3) \leq b_3 Q_3$$

Determine the most economical efficiency for the four plants using the following data (the fraction of BOD removed by decomposition is 6% for all four reaches):

	Reach 1-2 ($i = 1$)	Reach 2-3 ($i = 2$)	Reach 2-3 ($i = 3$)	Reach 3-4 ($i = 4$)
Q_i (gal/hr)	215,000	220,000	200,000	210,000
p_i (lb/hr)	500	3,000	6,000	1,000
b_i (lb BOD/gal)	.00085	.0009	.0008	.0008
Treatment cost ($/lb BOD removed)	.20	.25	.15	.18

11. *Loading Structure.* (Stark and Nicholes, 1972) The overhead crane with two lifting yokes in Figure 2.11 is used to transport mixed concrete to a yard for casting concrete barriers. The concrete bucket hangs at midpoint from the yoke. The crane end rails can support a maximum of 25 kip each and the yoke cables have a 20-kip capacity each. Determine the maximum load capacity, W_1 and W_2. (*Hint:* At equilibrium, the sum of moments about any point on the girder or yoke is zero.)

12. *Allocation of Aircraft to Routes.* Consider the problem of assigning aircraft to four routes according to the following data:

Aircraft type	Capacity (passengers)	Number of aircraft	Number of daily trips on route 1	2	3	4
1	50	5	3	2	2	1
2	30	8	4	3	3	2
3	20	10	5	5	4	2
Daily number of customers			1000	2000	900	1200

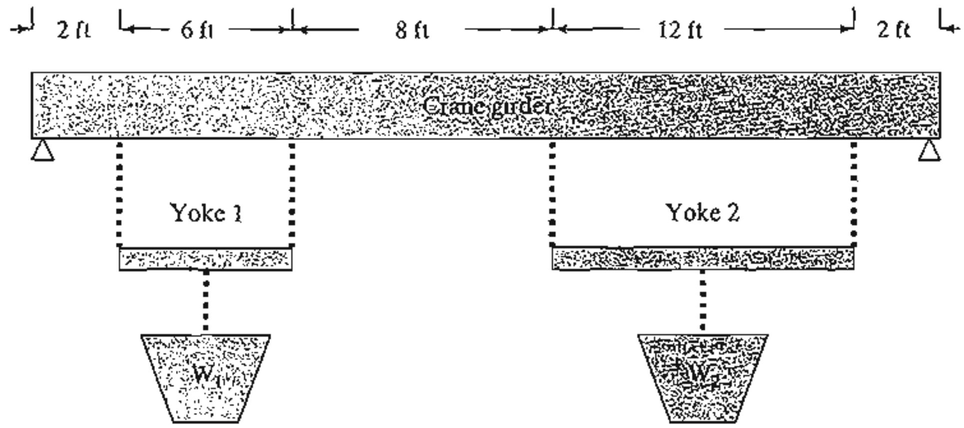


FIGURE 2.11
Overhead crane with two yokes (Problem 11)

The associated costs, including the penalties for losing customers because of space unavailability, are

	Operating cost ($) per trip on route			
Aircraft type	*1*	*2*	*3*	*4*
1	1000	1100	1200	1500
2	800	900	1000	1000
3	600	800	800	900
Penalty ($) per lost customer	40	50	45	70

Determine the optimum allocation of aircraft to routes and determine the associated number of trips.

2.4 COMPUTER SOLUTION WITH SOLVER AND AMPL

In practice, where typical linear programming models may involve thousands of variables and constraints, the only feasible way to solve such models is to use the computer. This section presents two distinct types of popular software: Excel Solver and AMPL. Solver is particularly appealing to spreadsheet users. AMPL is an algebraic modeling language that, like any other programming language, requires more expertise. Nevertheless, AMPL, and other similar languages,[3] offer great flexibility in modeling and executing large and complex LP models. Although the presentation in this section concentrates on LPs, both AMPL and Solver can be used with integer and nonlinear programs, as will be shown later in the book.

[3]Other known commercial packages include AIMMS, GAMS, LINGO, MPL, OPL Studio, and Xpress-Mosel.

2.4.1 LP Solution with Excel Solver

In Excel Solver, the spreadsheet is the input and output medium for the LP. Figure 2.12 shows the layout of the data for the Reddy Mikks model (file solverRM1.xls). The top of the figure includes four types of information: (1) input data cells (shaded areas,

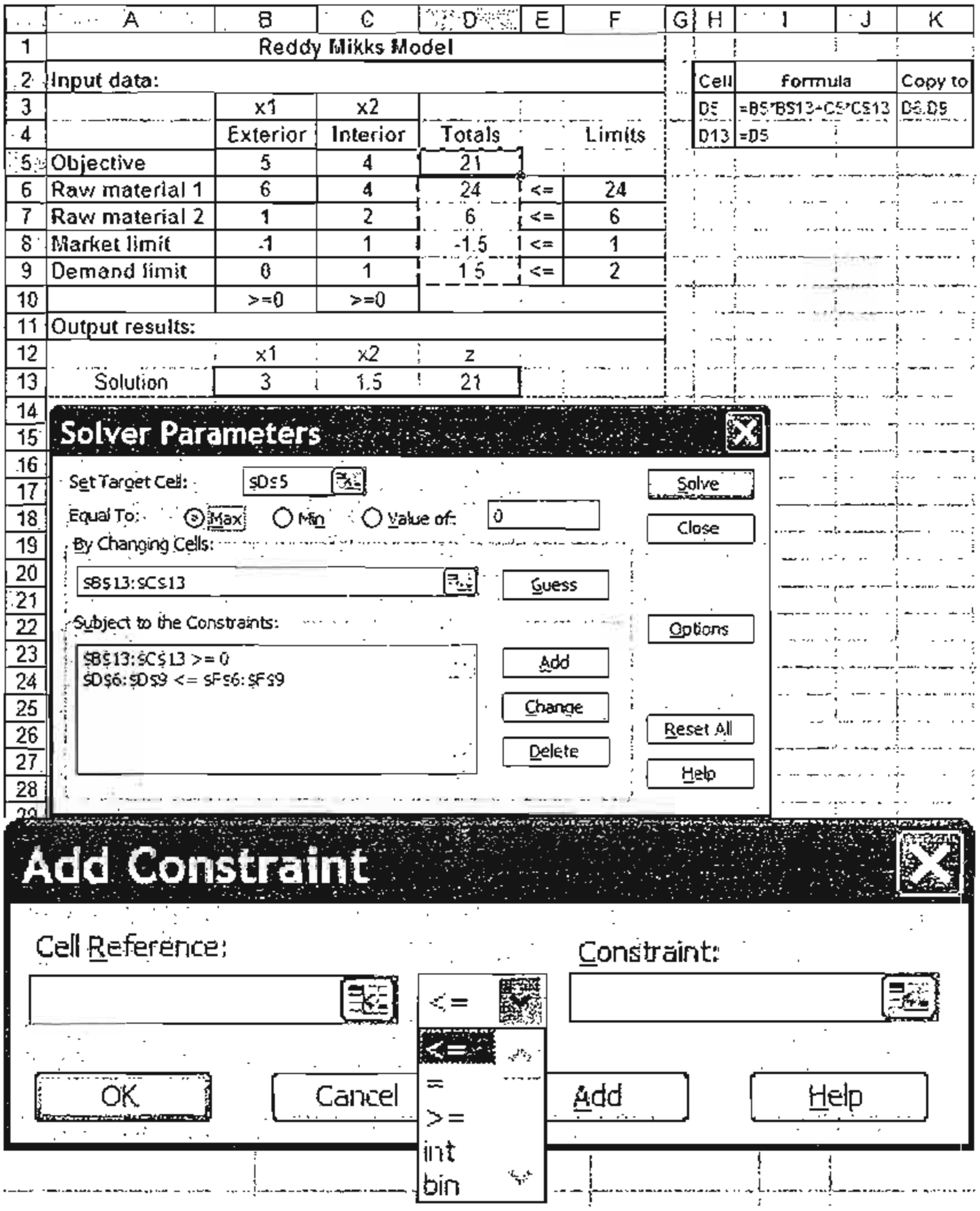


FIGURE 2.12

Defining the Reddy Mikks model with Excel Solver (file solverRM1.xls)

B5:C9 and F6:F9), (2) cells representing the variables and the objective function we seek to evaluate (solid rectangle cells, B13:D13), (3) algebraic definitions of the objective function and the left-hand side of the constraints (dashed rectangle cells, D5:D9), and (4) cells that provides explanatory names or symbols. Solver requires the first three types only. The fourth type enhances the readability of the model and serves no other purpose. The relative positioning of the four types of information on the spreadsheet need not follow the layout shown in Figure 2.12. For example, the cells defining the objective function and the variables need not be contiguous, nor do they have to be placed below the problem. What is important is that we know where they are so they can be referenced by Solver. Nonetheless, it is a good idea to use a format similar to the one suggested in Figure 2.12, because it makes the model more readable.

How does Solver link to the spreadsheet data? First we provide equivalent "algebraic" definitions of the objective function and the left-hand side of the constraints using the input data (shaded cells B5:C9 and F6:F9) and the objective function and variables (solid rectangle cells B13:D13), and then we place the resulting formulas in the appropriate cells of the dashed rectangle D5:D9. The following table shows the original LP functions and their placement in the appropriate cells:

	Algebraic expression	Spreadsheet formula	Entered in cell
Objective, z	$5x_1 + 4x_2$	=B5*$B$13+C5*$C$13	D5
Constraint 1	$6x_1 + 4x_2$	=B6*$B$13+C6*$C$13	D6
Constraint 2	$x_1 + 2x_2$	=B7*$B$13+C7*$C$13	D7
Constraint 3	$-x_1 + x_2$	=B8*$B$13+C8*$C$13	D8
Constraint 4	$0x_1 + x_2$	=B9*$B$13+C9*$C$13	D9

Actually, you only need to enter the formula for cell D5 and then copy it into cells D6:D9. To do so correctly, the fixed references $B$13 and $C$13 representing x_1 and x_2 must be used. For larger linear programs, it is more efficient to enter

=SUMPRODUCT(B5:C5,$B$13:$C$13)

in cell D5 and copy it into cells D6:D9.

All the elements of the LP model are now ready to be linked with Solver. From Excel's Tools menu, select Solver[4] to open the **Solver Parameters** dialogue box shown in the middle of Figure 2.12. First, you define the objective function, z, and the sense of optimization by entering the following data:

Set Target Cell: $D$5
Equal To: ⊙ Max
By Changing Cells: $B$13:$C$13

This information tells Solver that the variables defined by cells $B$13 and $C$13 are determined by maximizing the objective function in cell $D$5.

[4]If Solver does not appear under Tools, click Add-ins in the same menu and check Solver Add-in, then click OK.

The next step is to set up the constraints of the problems by clicking Add in the **Solver Parameters** dialogue box. The **Add Constraint** dialogue box will be displayed (see the bottom of Figure 2.12) to facilitate entering the elements of the constraints (left-hand side, inequality type, and right-hand side) as[5]

$D$6:$D$9<=$F$6:$F$9

A convenient substitute to typing in the cell ranges is to highlight cells D6:D9 to enter the left-hand sides and then cells F6:F9 to enter the right-hand sides. The same procedure can be used with Target Cell.

The only remaining constraints are the nonnegativity restrictions, which are added to the model by clicking Add in the **Add Constraint** dialogue box to enter

$B$13:$C$13>=0

Another way to enter the nonnegative constraints is to click Options on the **Solver Parameters** dialogue box to access the **Solver Options** dialogue box (see Figure 2.13) and then check ☑ Assume Non-Negative. While you are in the **Solver Options** box, you also need to check ☑ Assume Linear Model.

In general, the remaining default settings in **Solver Options** need not be changed. However, the default precision of .000001 may be set too "high" for some problems, and Solver may return the message "Solver could not find a feasible solution" when in fact the problem does have a feasible solution. In such cases, the precision needs to be adjusted to reflect less precision. If the same message persists, then the problem may be infeasible.

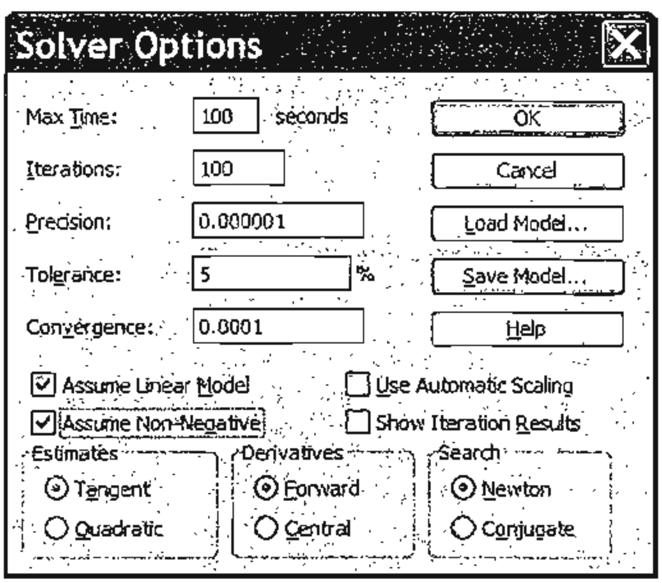


FIGURE 2.13
Solver options dialogue box

[5]You will notice that in the **Add Constraint** dialogue box (Figure 2.12), the middle box specifying the type of inequalities (<= and >=) has two additional options, int and bin, which stand for **integer** and **binary** and can be used with integer programs to restrict variables to integer or binary values (see Chapter 9).

For readability, you can use descriptive Excel range names instead of cell names. A range is created by highlighting the desired cells, typing the range name in the top left box of the sheet, and then pressing Return. Figure 2.14 (file solverRM2.xls) provides the details with a summary of the range names used in the model. You should contrast file solverRM2.xls with file solverRM1.xls to see how ranges are used in the formulas.

To solve the problem, click Solve on **Solver Parameters** (Figure 2.14). A new dialogue box, **Solver Results**, will then give the status of the solution. If the model setup is correct, the optimum value of z will appear in cell D5 and the values of x_1 and x_2 will go to cells B13 and C13, respectively. For convenience, we use cell D13 to exhibit the optimum value of z by entering the formula =D5 in cell D13 to display the entire optimum solution in contiguous cells.

If a problem has no feasible solution, Solver will issue the explicit message "Solver could not find a feasible solution." If the optimal objective value is unbounded, Solver will issue the somewhat ambiguous message "The Set Cell values do not converge." In either case, the message indicates that there is something wrong with the formulation of the model, as will be discussed in Section 3.5.

The **Solver Results** dialogue box will give you the opportunity to request further details about the solution, including the important sensitivity analysis report. We will discuss these additional results in Section 3.6.4.

The solution of the Reddy Mikks by Solver is straightforward. Other models may require a "bit of ingenuity" before they can be defined in a convenient manner. A class

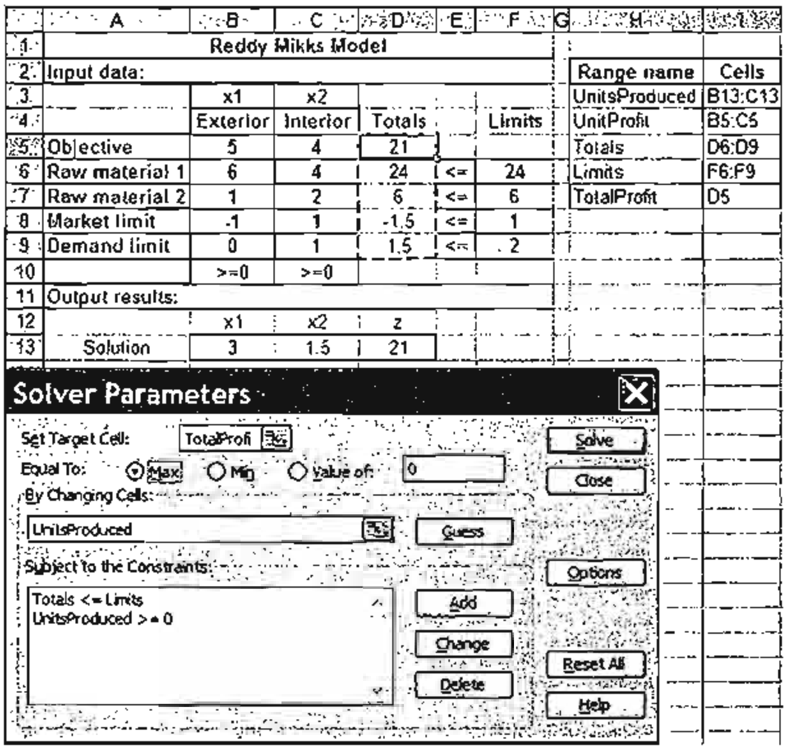


FIGURE 2.14

Use of range names in Excel Solver (file solverRM2.xls) of LP models that falls in this category deals with network optimization, as will be demonstrated in Chapter 6.

PROBLEM SET 2.4A

1. Modify the Reddy Mikks Solver model of Figure 2.12 to account for a third type of paint named "marine." Requirements per ton of raw materials 1 and 2 are .5 and .75 ton, respectively. The daily demand for the new paint lies between .5 ton and 1.5 tons. The profit per ton is $3.5 (thousand).
2. Develop the Excel Solver model for the following problems:
   - **(a)** The diet model of Example 2.2-2.
   - **(b)** Problem 16, Set 2.2a
   - **(c)** The urban renewal model of Example 2.3-1.
   - ***(d)** The currency arbitrage model of Example 2.3-2. (Hint: You will find it convenient to use the entire currency conversion matrix rather than the top diagonal elements only. Of course, you generate the bottom diagonal elements by using appropriate Excel formulas.)
   - **(e)** The multi-period production-inventory model of Example 2.3-5.

2.4.2 LP Solution with AMPL[6]

This section provides a brief introduction to AMPL. The material in Appendix A provides detailed coverage of AMPL syntax and will be cross-referenced opportunely with the presentation in this section as well as with other AMPL presentations throughout the book.

Four examples are presented here: The first two deal with the basics of AMPL, and the remaining two demonstrate more advanced usages to make a case for the advantages of AMPL.

Reddy Mikks Problem—a Rudimentary Model. AMPL provides a facility for modeling an LP in a rudimentary long-hand format. Figure 2.15 gives the self-explanatory code

```
var x1 >=0;
var x2 >=0;
maximize z: 5*x1+4*x2;
subject to
  c1: 6*x1+4*x2<=24;
  c2: x1+2*x2<=6;
  c3: -x1+x2<=1;
  c4: x2<=2;
solve;
display z,x1,x2;
```

Figure 2.15
Rudimentary AMPL model for the Reddy Mikks problem (file amplRM1.txt)

[6]For convenience, the AMPL student version, provided by AMPL Optimization LLC with instructions, is on the accompanying CD. Future updates may be downloaded from *www.ampl.com*. AMPL uses line commands and operates in a DOS (rather than Windows) environment. A recent beta version of a Windows interface can be found in *www.OptiRisk-Systems.com*.

for the Reddy Mikks model (file amplRM1.txt). All reserved keywords are in bold. All other names are user generated. The objective function and each of the constraints must be given a distinct user-generated name followed by a colon. Each statement closes with a semi-colon.

This rudimentary AMPL model is too specific in the sense that it requires developing a new code each time the data of the problem are changed. For practical problems with hundreds (even thousands) of variables and constraints, this long-hand format is cumbersome. AMPL alleviates this difficulty by dividing the problem into two components: (1) A general model that expresses the problem algebraically for any desired number of variables and constraints, and (2) specific data that drive the algebraic model. We will use the Reddy Mikks model to demonstrate the basic ideas of AMPL.

Reddy Mikks Problem—an Algebraic Model. Figure 2.16 lists the statements of the model (file amplRM2.txt). The file must be strictly text (ASCII). Comments are preceded with # and may appear anywhere in the model. The language is case sensitive and all its keywords (with few exceptions) must be in lower case. (Section A.2 provides more details.)

```
#----------------------------------------algebraic model
param m;
param n;
param c{1..n};
param b{1..m};
param a{1..m,1..n};

var x{1..n}>=0;

maximize z: sum{j in 1..n}c[j]*x[j];
subject to restr{i in 1..m}:
           sum{j in 1..n}a[i,j]*x[j]<=b[i];
#---------------------------------------specify model data
data;
param n:=2;
param m:=4;
param c:=1 5 2 4;
param b:=1 24  2 6  3 1  4 2;
param a:     1   2 :=
        1    6   4
        2    1   2
        3   -1   1
        4    0   1;
#---------------------------------------solve the problem
solve;
display z, x;
```

FIGURE 2.16

AMPL model of the Reddy Mikks problem with input data (file amplRM2.txt)

The algebraic model in AMPL views the general Reddy Mikks problem in the following generic format

$$\textbf{Maximize } z\text{: } \sum_{j=1}^{n} c_j x_j$$

$$\textbf{subject to } restr_i\text{: } \sum_{j=1}^{n} a_{ij} x_j \le b_i, i = 1, 2, \ldots, m$$

$$x_j \ge 0, j = 1, 2, \ldots, n$$

It assumes that the problem has n variables and m constraints. It gives the objective function and constraint i the (arbitrary) names z and $restr_i$. The rest of the parameters c_j, b_i, and a_{ij} are self-explanatory.

The model starts with the **param** statements that declare m, n, c, b, and a_{ij} as parameters (or constants) whose specific values are given in the input data section of the model. It translates $c_j (j = 1, 2, \ldots, n)$ as `c{1..n}`, $b_i (i = 1, 2, \ldots, m)$ as `b{1..m}`, and $a_{ij} (i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$ as `a{1..m,1..n}`. Next, the variables $x_j (j = 1, 2, \ldots, n)$ together with the nonnegativity restriction are defined by the **var** statement

```
var x{1..n}>=0;
```

If `>=0` is removed from the definition of x_j, then the variable is assumed unrestricted. The notation in `{}` represents the *set* of subscripts over which a `param` or a `var` is defined.

After defining all the parameters and the variables, we can develop the model itself. The objective function and constraints must each carry a distinct user-defined name followed by a colon (:). In the Reddy Mikks model the objective is given the name `z:` preceded by `maximize`, as the following AMPL statement states:

```
maximize z: sum{j in 1..n}c[j]*x[j];
```

The statement is a direct translation of maximize $z = \sum_{j=1}^{n} c_j x_j$ (with = replaced by :). Note the use of the brackets `[]` for representing the subscripts.

Constraint i is given the *root* name `restr` indexed over the set `{1..m}`:

```
restr{i in 1..m}:sum{j in 1..n}a[i,j]*x[j]<=b[i];
```

The statement is a direct translation of $\sum_{j=1}^{n} a_{ij} x_j \le b_i$. The keywords `subject to` are optional. This general model may now be used to solve any problem with any set of input data representing any number of constraints `m` and any number of variables `n`.

The `data;` section allows tailoring the model to the specific Reddy Mikks problem. Thus, `param n:=2;` and `param m:=4;` tell AMPL that the problem has 2 variables and 4 constraints. Note that the compound operator `:=` must be used and that the statement must start with the keyword param. For the single-subscripted parameter c, each element is represented by the subscript j followed by c_j separated by a blank space. Thus, the two values $c_1 = 5$ and $c_2 = 4$ translate to

```
param c:= 1 5  2  4;
```

The data for parameter b are entered in a similar manner.

For the double-subscripted parameter a, the top line defines the subscript j, and the subscript i is entered at the start of each row as

```
param a:   1    2 :=
       1   6    4
       2   1    2
       3  -1    1
       4   0    1;
```

In effect, the data a_{ij} read as a two-dimensional matrix with its rows designating i and its columns designating j. Note that a semicolon is needed only at the end of all a_{ij} data.

The model and its data are now ready. The command solve; invokes the solution and the command display z, x; provides the solution.

To execute the model, first invoke AMPL (by clicking ampl.exe in the AMPL directory). At the ampl prompt, enter the following model command, then press Return:

```
ampl: model AmplRM2.txt;
```

The output of the system will then appear on the screen as follows:

```
MINOS 5.5: Optimal solution found.
2 iterations, objective = 21

z = 21
x[*]:=

1 = 3
2 = 1.5
```

The bottom four lines are the result of executing display z,x;.

Actually, AMPL allows separating the algebraic model and the data into two independent files. This arrangement is advisable because once the model has been developed, only the data file needs to be changed. (See the end of Section A.2 for details.) In this book, we elect not to separate the model and data files, mainly for reasons of compactness.

The Arbitrage Problem. The simple Reddy Mikks model introduces some of the basic elements of AMPL. The more complex arbitrage model of Example 2.3-2 offers the opportunity to introduce additional AMPL capabilities that include: (1) imposing conditions on the elements of a set, (2) use of if then else to represent conditional values, (3) use of computed parameters, and (4) use of a simple print statement to retrieve output. These points are also discussed in more detail in Appendix A.

```
param inCurrency;                      #initial amount I
param outCurrency;                     #maximized holding y
param n;                               #nbr of currencies
param r{i in 1..n,j in 1..n:i<=j};#above-diagonal rates
param I;                               #initial amt of inCurrency
param maxTransaction{1..n};            #limit on transaction amt

var x{i in 1..n,j in 1..n}>=0;         #amt of i converted to j
var y>=0;                              #max amt of outCurrency

maximize z: y;
subject to
 r1{i in 1..n,j in 1..n}:x[i,j]<=maxTransaction[i];
 r2{i in 1..n}:(if i=inCurrency then I else 0)+
   sum{k in 1..n}(if k<i then r[k,i] else 1/r[i,k])*x[k,i]=
     (if i=outCurrency then y else 0)+sum{j in 1..n}x[i,j];
#---------------------------------------input data
data;
param inCurrency=1;
param outCurrency=1;
param n:=5;
#                     $      euro   pound    yen    KD
param r:              1       2       3       4      5:=
               1      1      .769    .625    105    .342   #$
               2      .      1       .813    137    .445   #euro
               3      .      .       1       169    .543   #pound
               4      .      .       .       1      .0032  #yen
               5      .      .       .       .      1;     #KD
param I:= 5;
param maxTransaction:=1 5 2 3 3 3.5 4 100 5 2.8;
#---------------------------------------Solution command
solve;
display z,y,x>file2.out;
print "rate of return =",trunc(100*(z-I)/I,4),"%">file2.out;
```

FIGURE 2.17

AMPL model of the Arbitrage problem (file amplEx2.3-2.txt)

Figure 2.17 (file amplEx2.3-2.txt) gives the AMPL code for the arbitrage problem. The model is general in the sense that it can be used to maximize the final holdings `y` of any currency, named `outCurrency`, starting with an initial amount `I` of another currency, named `inCurrency`. Additionally, any number of currencies, `n`, can be involved in the arbitrage process.

The exchange rates are defined as

```
param r{i in 1..n,j in 1..n:i<=j};
```

The definition gives only the diagonal and above-diagonal elements by imposing the condition `i<=j` (preceded by a colon) on the set `{i in 1..n,j in 1..n}`. With this definition, reciprocals are used to compute the below-diagonal rates, as will be shown shortly.

The variable x_{ij}, representing the amount of currency i converted to currency j, is defined as

```
var x{i in 1..n,j in 1..n}>=0;
```

The model has two sets of constraints: The first set with the root name `r1` sets the limits on the amounts of any currency conversion transaction by using the statement

```
r1{i in 1..n,j in 1..n}: x[i,j]<=maxTransaction[i];
```

The second set of constraints with the root name `r2` is a translation of the restriction

$$(\text{Input to currency } i) = (\text{Output from currency } i)$$

Its statement is given as

```
r2{i in 1..n}:
  (if i=inCurrency then I else 0)+
   sum{k in 1..n}(if k<i then r[k,i] else 1/r[i,k])*x[k,i]
      =(if i=outCurrency then y else 0)+sum{j in 1..n}x[i,j];
```

This type of constraints is ideal for the use of the special construct `if then else` to specify conditional values. In the left-hand side of the constraint, the expression

```
(if i=inCurrency then I else 0)
```

says that in the constraint for the input currency `(i=inCurrency)` there is an external input `I`, else the external input is zero. Next, the expression

```
sum{k in 1..n}(if k<i then r[k,i] else 1/r[i,k])*x[k,i]
```

computes the input funds from other currency converted to the input currency. If you review Example 2.3-2 you will notice that when `k<i`, the conversion uses the above-diagonal elements of the exchange rate `r`. Otherwise, the *row* reciprocal is used for the below-diagonal elements (diagonal elements are 1). This is precisely what `if then else` does. (See Section A.3 for details.)

The `if`-expression in the right-hand side of constraint `r2` can be explained in a similar manner—namely,

```
(if i=outCurrency then y else 0)
```

says that the external output is `y` for `outCurrency` and zero for all others.

We can enhance the readability of constraints `r2` by defining the following **computed parameter** (see Section A.3) that defines the entire exchange rate table:

```
Param rate{k in 1..n,i in 1..n}
         =(if k<i then r[k,i] else 1/r[i,k])
```

In this case, constraints `r2` become

```
r2{i in 1..n}:
  (if i=inCurrency then I else 0)+ sum{k in 1..n}rate[k,i]*x[k,i]
      =(if i=outCurrency then y else 0)+sum{j in 1..n}x[i,j];
```

In the `data;` section, `inCurrency` and `outCurrency` each equal 1, which means that the problem is seeking the maximum dollar output using an initial amount of $5 million. In general, `inCurrency` and `outCurrency` may designate any distinct currencies. For example, setting `inCurrency` equal to `2` and `outCurrency` equal to 4 maximizes the yen output given a 5 million euros initial investment.

The unspecified entries of `param r` are flagged in AMPL with dots (.). These values are then overridden either by using the reciprocal as shown in Figure 2.17 or through the use of the computed parameter `rate` as shown above. The alternative to using dots is to unnecessarily compute and enter the below-diagonal elements as data.

The `display` statement sends the output to file file2.out instead of defaulting it to the screen. The `print` statement computes and truncates the rate of return and sends the output to file file2.out. The `print` statement can also be formatted using `printf`, just as in any higher level programming language. (See Section A.5.2 for details.)

It is important to notice that input data in AMPL need not be hard-coded in the model, as they can be retrieved from external files, spreadsheets, and databases (see Section A.5 for details). This is crucial in the arbitrage model, where the volatile exchange rates must often be accepted within less than 10 seconds. By allowing the AMPL model to receive its data from a database that automatically updates the exchange rates, the model can provide timely optimal solutions.

The Bus Scheduling Problem. The bus scheduling problem of Example 2.3-8 provides an interesting modeling situation in AMPL. Of course, we can always use a two-subscripted parameter, similar to parameter `a` in the Reddy Mikks model in Section 2.4.2 (Figure 2.16), but this may be cumbersome in this case. Instead, we can take advantage of the special structure of the constraints and use conditional expressions to represent them implicitly.

The left-hand side of constraint 1 is $x_1 + x_m$, where m is the total number of periods in a 24-hour day (= 6 in the present example). For the remaining constraints, the left-hand side takes the form $x_{i-1} + x_i, i = 2, 3, \ldots, m$. Using `if then else` (as we did in the arbitrage problem), all m constraints can be represented compactly by one statement as shown in Figure 2.18 (file amplEx2.3-8.txt). This representation is superior to defining the left-hand side of the constraints as an explicit parameter.

AMPL offers a wide range of programming capabilities. For example, the input/output data can be secured from/sent to external files, spreadsheets, and databases and the model can be executed interactively for a wide variety of options that allow testing different scenarios. The details are given in Appendix A. Also, many AMPL models are presented throughout the book with cross references to the material in Appendix A to assist you in understanding these options.

```
param m;
param  min_nbr_buses{1..m};
var x_nbr_buses{1..m} >= 0;
minimize tot_nbr_buses: sum {i in 1..m} x_nbr_buses[i];
subject to constr_nbr{i in 1..m}:
        if i=1 then
              x_nbr_buses[i]+x_nbr_buses[m]
        else
              x_nbr_buses[i-1]+x_nbr_buses[i] >= min_nbr_buses[i];

data;
param m:=6;
param min_nbr_buses:= 1 4   2 8   3 10   4 7   5 12   6 4;

solve;
display tot_nbr_buses, x_nbr_buses;
```

FIGURE 2.18

AMPL model of the bus scheduling problem of Example 2.3-8 (file amplEx2.3-8.txt)

PROBLEM SET 2.4B

1. In the Reddy Mikks model, suppose that a third type of paint, named "marine," is produced. The requirements per ton of raw materials $M1$ and $M2$ are .5 and .75 ton, respectively. The daily demand for the new paint lies between .5 ton and 1.5 tons and the profit per ton is \$3.5 (thousand). Modify the Excel Solver model solverRM2.xls and the AMPL model amplRM2.txt to account for the new situation and determine the optimum solution. Compare the additional effort associated with each modification.
2. Develop AMPL models for the following problems:
   - **(a)** The diet problem of Example 2.2-2 and find the optimum solution.
   - **(b)** Problem 4, Set 2.3b.
   - ***(c)** Problem 7, Set 2.3d.
   - **(d)** Problem 7, Set 2.3g.
   - **(e)** Problem 9, Set 2.3g.
   - ***(f)** Problem 10, Set 2.3g.

REFERENCES

Fourer, R., D. Gay, and B. Kernighan, *AMPL, A Modeling Language for Mathematical Programming*, 2nd ed., Brooks/Cole-Thomson, Pacific Grove, CA, 2003.

Kornbluth, J., and G. Salkin, *The Management for Corporate Financial Assets: Applications of Mathematical Programming Models*, Academic Press, London, 1987.

Shepard, R., D. Hartley, P. Hasman, L. Thorpe, and M. Bathe, *Applied Operations Research*, Plenum Press, New York, 1988.

Stark, R., and R. Nicholes, *Mathematical Programming Foundations for Design: Civil Engineering Systems*, McGraw-Hill, New York, 1972.

Laidlaw, C. *Linear Programming for Urban Development Plan Evaluation*, Praegers, London, 1972.

Lewis, T., "Personal Operations Research: Practicing OR on Ourselves," *Interfaces*, Vol. 26, No. 5, pp. 34–41, 1996.

William, H., *Model Building in Mathematical Programming*, 4th ed., Wiley, New York, 1994.

CHAPTER 3

The Simplex Method and Sensitivity Analysis

Chapter Guide. This chapter details the simplex method for solving the general LP problem. It also explains how simplex-based sensitivity analysis is used to provide important economic interpretations about the optimum solution, including the *dual prices* and the *reduced cost.*

The simplex method computations are particularly tedious, repetitive, and, above all, boring. As you do these computations, you should not lose track of the big picture; namely, the simplex method attempts to move from one corner point of the solution space to a better corner point until the optimum is found. To assist you in this regard, TORA's interactive *user-guided* module (with instant feedback) allows you to decide how the computations should proceed while relieving you of the burden of the tedious computations. In this manner, you get to understand the concepts without being overwhelmed by the computational details. Rest assured that once you have learned how the simplex method works (and it is important that you do understand the concepts), computers will carry out the tedious work and you will *never* again need to solve an LP manually.

Throughout my teaching experience, I have noticed that while students can easily carry out the tedious simplex method computations, in the end, some cannot tell why they are doing them or what the solution is. To assist in overcoming this potential difficulty, the material in the chapter stresses the interpretation of each iteration in terms of the solution to the original problem.

When you complete the material in this chapter, you will be in a position to read and interpret the output reports provided by commercial software. The last section describes how these reports are generated in AMPL, Excel Solver, and TORA.

This chapter includes a summary of 1 real-life application, 11 solved examples, 1 AMPL model, 1 Solver model, 1 TORA model, 107 end-of-section problems, and 3 cases. The cases are in Appendix E on the CD. The AMPL/Excel/Solver/TORA programs are in folder ch3Files.

Real Life Application—Optimization of Heart Valve Production

Biological heart valves in different sizes are bioprostheses manufactured from porcine hearts for human implantation. On the supply side, porcine hearts cannot be "produced" to specific sizes. Moreover, the exact size of a manufactured valve cannot be determined until the biological component of pig heart has been processed. As a result, some sizes may be overstocked and others understocked. A linear programming model was developed to reduce overstocked sizes and increase understocked sizes. The resulting savings exceeded $1,476,000 in 1981, the year the study was made. The details of this study are presented in Case 2, Chapter 24 on the CD.

3.1 LP MODEL IN EQUATION FORM

The development of the simplex method computations is facilitated by imposing two requirements on the constraints of the problem:

1. All the constraints (with the exception of the nonnegativity of the variables) are equations with nonnegative right-hand side.
2. All the variables are nonnegative.

These two requirements are imposed here primarily to standardize and streamline the simplex method calculations. It is important to know that all commercial packages (and TORA) directly accept inequality constraints, nonnegative right-hand side, and unrestricted variables. Any necessary preconditioning of the model is done internally in the software before the simplex method solves the problem.

3.1.1 Converting Inequalities into Equations with Nonnegative Right-Hand Side

In ($\leq$) constraints, the right-hand side can be thought of as representing the limit on the availability of a resource, in which case the left-hand side would represent the usage of this limited resource by the activities (variables) of the model. The difference between the right-hand side and the left-hand side of the ($\leq$) constraint thus yields the *unused* or *slack* amount of the resource.

To convert a ($\leq$)-inequality to an equation, a nonnegative **slack variable** is added to the left-hand side of the constraint. For example, in the Reddy Mikks model (Example 2.1-1), the constraint associated with the use of raw material $M1$ is given as

$$6x_1 + 4x_2 \leq 24$$

Defining s_1 as the slack or unused amount of $M1$, the constraint can be converted to the following equation:

$$6x_1 + 4x_2 + s_1 = 24, s_1 \geq 0$$

Next, a ($\geq$)-constraint sets a lower limit on the activities of the LP model, so that the amount by which the left-hand side exceeds the minimum limit represents a *surplus*. The conversion from ($\geq$) to ($=$) is achieved by subtracting a nonnegative

surplus variable from the left-hand side of the inequality. For example, in the diet model (Example 2.2-2), the constraint representing the minimum feed requirements is

$$x_1 + x_2 \geq 800$$

Defining S_1 as the surplus variable, the constraint can be converted to the following equation

$$x_1 + x_2 - S_1 = 800, S_1 \geq 0$$

The only remaining requirement is for the right-hand side of the resulting equation to be nonnegative. The condition can always be satisfied by multiplying both sides of the resulting equation by -1, where necessary. For example, the constraint

$$-x_1 + x_2 \leq -3$$

is equivalent to the equation

$$-x_1 + x_2 + s_1 = -3, s_1 \geq 0$$

Now, multiplying both sides by -1 will render a nonnegative right-hand side, as desired—that is,

$$x_1 - x_2 - s_1 = 3$$

PROBLEM SET 3.1A

*1. In the Reddy Mikks model (Example 2.2-1), consider the feasible solution $x_1 = 3$ tons and $x_2 = 1$ ton. Determine the value of the associated slacks for raw materials *M*1 and *M*2.

2. In the diet model (Example 2.2-2), determine the surplus amount of feed consisting of 500 lb of corn and 600 lb of soybean meal.

3. Consider the following inequality

$$10x_1 - 3x_2 \geq -5$$

Show that multiplying both sides of the inequality by -1 and then converting the resulting inequality into an equation is the same as converting it first to an equation and then multiplying both sides by -1.

*4. Two different products, *P*1 and *P*2, can be manufactured by one or both of two different machines, *M*1 and *M*2. The unit processing time of either product on either machine is the same. The daily capacity of machine *M*1 is 200 units (of either *P*1 or *P*2, or a mixture of both) and the daily capacity of machine *M*2 is 250 units. The shop supervisor wants to balance the production schedule of the two machines such that the total number of units produced on one machine is within 5 units of the number produced on the other. The profit per unit of *P*1 is \$10 and that of *P*2 is \$15. Set up the problem as an LP in equation form.

5. Show how the following objective function can be presented in equation form:

$$\text{Minimize } z = \max\{|x_1 - x_2 + 3x_3|, |-x_1 + 3x_2 - x_3|\}$$

$$x_1, x_2, x_3 \geq 0$$

(*Hint*: $|a| \leq b$ is equivalent to $a \leq b$ and $a \geq -b$.)

6. Show that the m equations:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, i = 1, 2, \ldots, m$$

are equivalent to the following $m + 1$ inequalities:

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\right)x_j \geq \sum_{i=1}^{m} b_i$$

3.1.2 Dealing with Unrestricted Variables

In Example 2.3-6 we presented a multiperiod production smoothing model in which the workforce at the start of each period is adjusted up or down depending on the demand for that period. Specifically, if x_i (≥ 0) is the workforce size in period i, then x_{i+1} (≥ 0) the workforce size in period $i + 1$ can be expressed as

$$x_{i+1} = x_i + y_{i+1}$$

The variable y_{i+1} must be unrestricted in sign to allow x_{i+1} to increase or decrease relative to x_i depending on whether workers are hired or fired, respectively.

As we will see shortly, the simplex method computations require all the variables be nonnegative. We can always account for this requirement by using the substitution

$$y_{i+1} = y_{i+1}^- - y_{i+1}^+, \text{ where } y_{i+1}^- \geq 0 \text{ and } y_{i+1}^+ \geq 0$$

To show how this substitution works, suppose that in period 1 the workforce is $x_1 = 20$ workers and that the workforce in period 2 will be increased by 5 to reach 25 workers. In terms of the variables y_2^- and y_2^+, this will be equivalent to $y_2^- = 5$ and $y_2^+ = 0$ or $y_2 = 5 - 0 = 5$. Similarly, if the workforce in period 2 is reduced to 16, then we have $y_2^- = 0$ and $y_2^+ = 4$, or $y_2 = 0 - 4 = -4$. The substitution also allows for the possibility of no change in the workforce by letting both variables assume a zero value.

You probably are wondering about the possibility that both y_2^- and y_2^+ may assume positive values simultaneously. Intuitively, as we explained in Example 2.3-6, this cannot happen, because it means that we can hire and fire a worker at the same time. This intuition is also supported by a mathematical proof that shows that, in any simplex method solution, it is impossible that both variables will assume positive values simultaneously.

PROBLEM SET 3.1B

1. McBurger fast-food restaurant sells quarter-pounders and cheeseburgers. A quarter-pounder uses a quarter of a pound of meat, and a cheeseburger uses only .2 lb. The restaurant starts the day with 200 lb of meat but may order more at an additional cost of 25 cents per pound to cover the delivery cost. Any surplus meat at the end of the day is donated to charity. McBurger's profits are 20 cents for a quarter-pounder and 15 cents for a cheeseburger. McBurger does not expect to sell more than 900 sandwiches in any one

day. How many of each type sandwich should McBurger plan for the day? Solve the problem using TORA, Solver, or AMPL.

2. Two products are manufactured in a machining center. The productions times per unit of products 1 and 2 are 10 and 12 minutes, respectively. The total regular machine time is 2500 minutes per day. In any one day, the manufacturer can produce between 150 and 200 units of product 1, but no more than 45 units of product 2. Overtime may be used to meet the demand at an additional cost of $.50 per minute. Assuming that the unit profits for products 1 and 2 are $6.00 and $7.50, respectively, formulate the problem as an LP model, then solve with TORA, Solver, or AMPL to determine the optimum production level for each product as well as any overtime needed in the center.

*3. JoShop manufactures three products whose unit profits are $2, $5, and $3, respectively. The company has budgeted 80 hours of labor time and 65 hours of machine time for the production of three products. The labor requirements per unit of products 1, 2, and 3 are 2, 1, and 2 hours, respectively. The corresponding machine-time requirements per unit are 1, 1, and 2 hours. JoShop regards the budgeted labor and machine hours as goals that may be exceeded, if necessary, but at the additional cost of $15 per labor hour and $10 per machine hour. Formulate the problem as an LP, and determine its optimum solution using TORA, Solver, or AMPL.

4. In an LP in which there are several unrestricted variables, a transformation of the type $x_j = x_j^- - x_j^+, x_j^-, x_j^+ \geq 0$ will double the corresponding number of nonnegative variables. We can, instead, replace k unrestricted variables with exactly $k + 1$ nonnegative variables by using the substitution $x_j = x_j' - w, x_j', w \geq 0$. Use TORA, Solver, or AMPL to show that the two methods produce the same solution for the following LP:

$$\text{Maximize } z = -2x_1 + 3x_2 - 2x_3$$

subject to

$$4x_1 - x_2 - 5x_3 = 10$$

$$2x_1 + 3x_2 + 2x_3 = 12$$

$$x_1 \geq 0, x_2, x_3 \text{ unrestricted}$$

3.2 TRANSITION FROM GRAPHICAL TO ALGEBRAIC SOLUTION

The ideas conveyed by the graphical LP solution in Section 2.2 lay the foundation for the development of the algebraic simplex method. Figure 3.1 draws a parallel between the two methods. In the graphical method, the solution space is delineated by the half-spaces representing the constraints, and in the simplex method the solution space is represented by m simultaneous linear equations and n nonnegative variables.

We can see visually why the graphical solution space has an infinite number of solution points, but how can we draw a similar conclusion from the algebraic representation of the solution space? The answer is that in the algebraic representation the number of equations m is always *less than or equal to* the number of variables n.[1] If $m = n$, and the equations are consistent, the system has only one solution; but if $m < n$ (which

[1]If the number of equations m is larger than the number of variables n, then at least $m - n$ equations must be redundant.

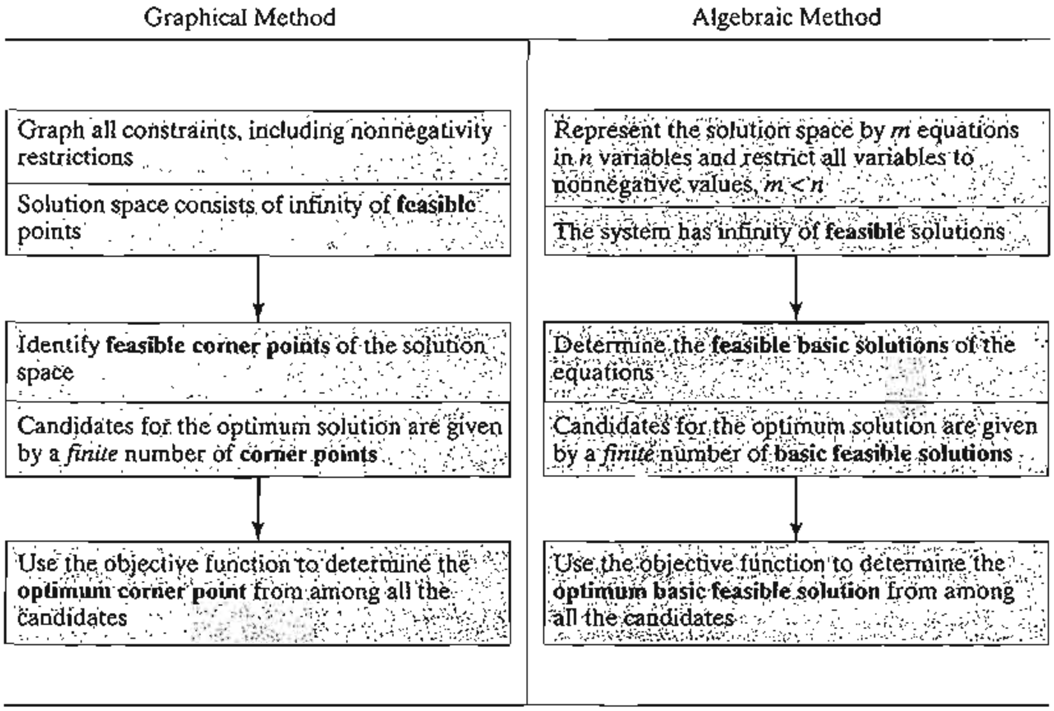


FIGURE 3.1

Transition from graphical to algebraic solution

represents the majority of LPs), then the system of equations, again if consistent, will yield an infinite number of solutions. To provide a simple illustration, the equation $x = 2$ has $m = n = 1$, and the solution is obviously unique. But, the equation $x + y = 1$ has $m = 1$ and $n = 2$, and it yields an infinite number of solutions (any point on the straight line $x + y = 1$ is a solution).

Having shown how the LP solution space is represented algebraically, the candidates for the optimum (i.e., corner points) are determined from the simultaneous linear equations in the following manner:

Algebraic Determination of Corner Points.

In a set of $m \times n$ equations ($m < n$), if we set $n - m$ variables equal to zero and then solve the m equations for the remaining m variables, the resulting solution, if unique, is called a **basic solution** and must correspond to a (feasible or infeasible) corner point of the solution space. This means that the *maximum* number of corner points is

$$C_m^n = \frac{n!}{m!(n-m)!}$$

The following example demonstrates the procedure.

Example 3.2-1

Consider the following LP with two variables:

$$\text{Maximize } z = 2x_1 + 3x_2$$

subject to

$$2x_1 + x_2 \leq 4$$
$$x_1 + 2x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

Figure 3.2 provides the graphical solution space for the problem.

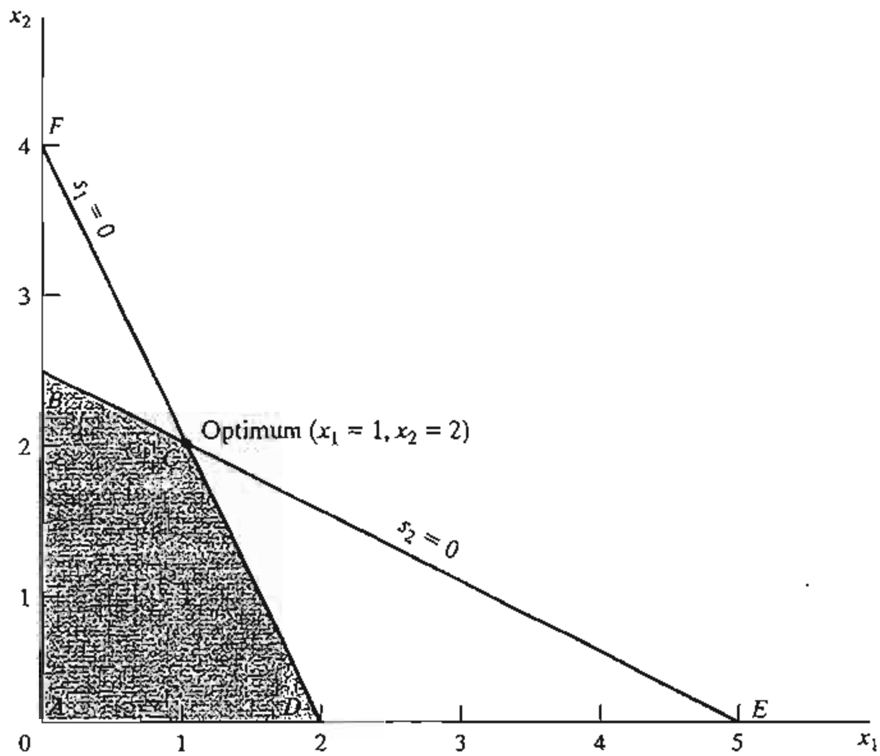
Algebraically, the solution space of the LP is represented as:

$$2x_1 + x_2 + s_1 = 4$$
$$x_1 + 2x_2 + s_2 = 5$$
$$x_1, x_2, s_1, s_2 \geq 0$$

The system has $m = 2$ equations and $n = 4$ variables. Thus, according to the given rule, the corner points can be determined algebraically by setting $n - m = 4 - 2 = 2$ variables equal to

FIGURE 3.2

LP solution space of Example 3.2-1



zero and then solving for the remaining $m = 2$ variables. For example, if we set $x_1 = 0$ and $x_2 = 0$, the equations provide the unique (basic) solution

$$s_1 = 4, s_2 = 5$$

This solution corresponds to point A in Figure 3.2 (convince yourself that $s_1 = 4$ and $s_2 = 5$ at point A). Another point can be determined by setting $s_1 = 0$ and $s_2 = 0$ and then solving the two equations

$$2x_1 + x_2 = 4$$

$$x_1 + 2x_2 = 5$$

This yields the basic solution ($x_1 = 1, x_2 = 2$), which is point C in Figure 3.2.

You probably are wondering how one can decide which $n - m$ variables should be set equal to zero to target a specific corner point. Without the benefit of the graphical solution (which is available only for two or three variables), we cannot say which $(n - m)$ zero variables are associated with which corner point. But that does not prevent us from enumerating *all* the corner points of the solution space. Simply consider *all* combinations in which $n - m$ variables are set to zero and solve the resulting equations. Once done, the optimum solution is the feasible basic solution (corner point) that yields the best objective value.

In the present example we have $C_2^4 = \frac{4!}{2!2!} = 6$ corner points. Looking at Figure 3.2, we can immediately spot the four corner points A, B, C, and D. Where, then, are the remaining two? In fact, points E and F also are corner points for the problem, but they are *infeasible* because they do not satisfy all the constraints. These infeasible corner points are not candidates for the optimum.

To summarize the transition from the graphical to the algebraic solution, the zero $n - m$ variables are known as **nonbasic variables**. The remaining m variables are called **basic variables** and their solution (obtained by solving the m equations) is referred to as **basic solution**. The following table provides all the basic and nonbasic solutions of the current example.

Nonbasic (zero) variables	Basic variables	Basic solution	Associated corner point	Feasible?	Objective value, z
(x_1, x_2)	(s_1, s_2)	(4, 5)	A	Yes	0
(x_1, s_1)	(x_2, s_2)	(4, −3)	F	No	—
(x_1, s_2)	(x_2, s_1)	(2.5, 1.5)	B	Yes	7.5
(x_2, s_1)	(x_1, s_2)	(2, 3)	D	Yes	4
(x_2, s_2)	(x_1, s_1)	(5, −6)	E	No	—
(s_1, s_2)	(x_1, x_2)	**(1, 2)**	C	**Yes**	**8 (optimum)**

Remarks. We can see from the computations above that as the problem size increases (that is, m and n become large), the procedure of enumerating all the corner points involves prohibitive computations. For example, for $m = 10$ and $n = 20$, it is necessary to solve $C_{10}^{20} = 184{,}756$ sets of 10×10 equations, a staggering task indeed, particularly when we realize that a (10×20)-LP is a small size in most real-life situations, where hundreds or even thousands of variables and constraints are not unusual. The simplex method alleviates this computational burden dramatically by investigating only a fraction of all possible basic feasible solutions (corner points) of the solution space. In essence, the simplex method utilizes an intelligent search procedure that locates the optimum corner point in an efficient manner.

PROBLEM SET 3.2A

1. Consider the following LP:

$$\text{Maximize } z = 2x_1 + 3x_2$$

subject to

$$x_1 + 3x_2 \le 6$$
$$3x_1 + 2x_2 \le 6$$
$$x_1, x_2 \ge 0$$

   (a) Express the problem in equation form.
   (b) Determine all the basic solutions of the problem, and classify them as feasible and infeasible.
   ***(c)** Use direct substitution in the objective function to determine the optimum basic feasible solution.
   (d) Verify graphically that the solution obtained in (c) is the optimum LP solution—hence, conclude that the optimum solution can be determined algebraically by considering the basic feasible solutions only.
   ***(e)** Show how the *infeasible* basic solutions are represented on the graphical solution space.

2. Determine the optimum solution for each of the following LPs by enumerating all the basic solutions.
   (a) Maximize $z = 2x_1 - 4x_2 + 5x_3 - 6x_4$
   subject to

$$x_1 + 4x_2 - 2x_3 + 8x_4 \le 2$$
$$-x_1 + 2x_2 + 3x_3 + 4x_4 \le 1$$
$$x_1, x_2, x_3, x_4 \ge 0$$

   (b) Minimize $z = x_1 + 2x_2 - 3x_3 - 2x_4$
   subject to

$$x_1 + 2x_2 - 3x_3 + x_4 = 4$$
$$x_1 + 2x_2 + x_3 + 2x_4 = 4$$
$$x_1, x_2, x_3, x_4 \ge 0$$

*3. Show algebraically that all the basic solutions of the following LP are infeasible.

$$\text{Maximize } z = x_1 + x_2$$

subject to

$$x_1 + 2x_2 \le 6$$
$$2x_1 + x_2 \le 16$$
$$x_1, x_2 \ge 0$$

4. Consider the following LP:

$$\text{Maximize } z = 2x_1 + 3x_2 + 5x_3$$

subject to

$$-6x_1 + 7x_2 - 9x_3 \geq 4$$
$$x_1 + x_2 + 4x_3 = 10$$
$$x_1, x_3 \geq 0$$
$$x_2 \text{ unrestricted}$$

Conversion to the equation form involves using the substitution $x_2 = x_2^- - x_2^+$. Show that a basic solution cannot include both x_2^- and x_2^+ simultaneously.

5. Consider the following LP:

$$\text{Maximize } z = x_1 + 3x_2$$

subject to

$$x_1 + x_2 \leq 2$$
$$-x_1 + x_2 \leq 4$$
$$x_1 \text{ unrestricted}$$
$$x_2 \geq 0$$

(a) Determine all the basic feasible solutions of the problem.
(b) Use direct substitution in the objective function to determine the best basic solution.
(c) Solve the problem graphically, and verify that the solution obtained in (c) is the optimum.

3.3 THE SIMPLEX METHOD

Rather than enumerating *all* the basic solutions (corner points) of the LP problem (as we did in Section 3.2), the simplex method investigates only a "select few" of these solutions. Section 3.3.1 describes the *iterative* nature of the method, and Section 3.3.2 provides the computational details of the simplex algorithm.

3.3.1 Iterative Nature of the Simplex Method

Figure 3.3 provides the solution space of the LP of Example 3.2-1. Normally, the simplex method starts at the origin (point A) where $x_1 = x_2 = 0$. At this starting point, the value of the objective function, z, is zero, and the logical question is whether an increase in nonbasic x_1 and/or x_2 above their current zero values can improve (increase) the value of z. We answer this question by investigating the objective function:

$$\text{Maximize } z = 2x_1 + 3x_2$$

The function shows that an increase in either x_1 or x_2 (or both) above their current zero values will *improve* the value of z. The design of the simplex method calls for increasing *one variable at a time*, with the selected variable being the one with the *largest*

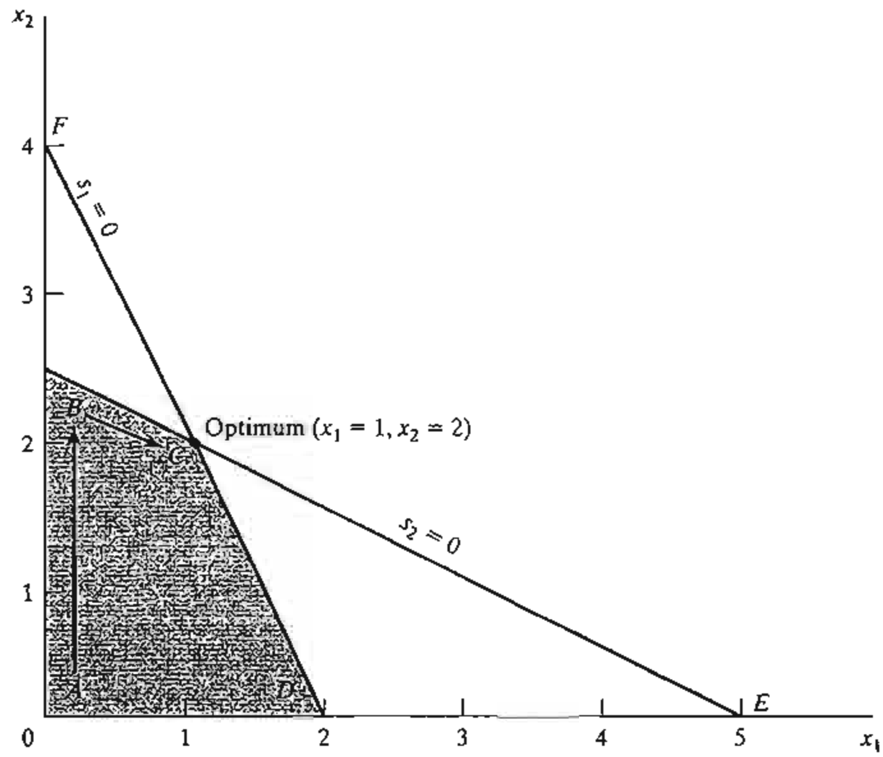


FIGURE 3.3
Iterative process of the simplex method

rate of improvement in z. In the present example, the value of z will increase by 2 for each unit increase in x_1 and by 3 for each unit increase in x_2. This means that the *rate* of improvement in the value of z is 2 for x_1 and 3 for x_2. We thus elect to increase x_2, the variable with the largest rate of improvement. Figure 3.3 shows that the value of x_2 must be increased until corner point B is reached (recall that stopping short of reaching corner point B is not optimal because a candidate for the optimum must be a corner point). At point B, the simplex method will then increase the value of x_1 to reach the improved corner point C, which is the optimum. The path of the simplex algorithm is thus defined as $A \rightarrow B \rightarrow C$. Each corner point along the path is associated with an **iteration**. It is important to note that the simplex method moves alongside the **edges** of the solution space, which means that the method cannot cut across the solution space, going from A to C directly.

We need to make the transition from the graphical solution to the algebraic solution by showing how the points A, B, and C are represented by their basic and nonbasic variables. The following table summarizes these representations:

Corner point	Basic variables	Nonbasic (zero) variables
A	s_1, s_2	x_1, x_2
B	s_1, x_2	x_1, s_2
C	x_1, x_2	s_1, s_2

Notice the change pattern in the basic and nonbasic variables as the solution moves along the path $A \rightarrow B \rightarrow C$. From A to B, nonbasic x_2 at A becomes basic at B and basic s_2 at A becomes nonbasic at B. In the terminology of the simplex method, we say that x_2 is the **entering variable** (because it enters the basic solution) and s_2 is the **leaving variable** (because it leaves the basic solution). In a similar manner, at point B, x_1 *enters* (the basic solution) and s_1 *leaves*, thus leading to point C.

PROBLEM SET 3.3A

1. In Figure 3.3, suppose that the objective function is changed to

$$\text{Maximize } z = 8x_1 + 4x_2$$

   Identify the path of the simplex method and the basic and nonbasic variables that define this path.
2. Consider the graphical solution of the Reddy Mikks model given in Figure 2.2. Identify the path of the simplex method and the basic and nonbasic variables that define this path.

*3. Consider the three-dimensional LP solution space in Figure 3.4, whose feasible extreme points are $A, B, \ldots,$ and J.
   - **(a)** Which of the following pairs of corner points cannot represent *successive* simplex iterations: (A, B), (B, D), (E, H), and (A, I)? Explain the reason.
   - **(b)** Suppose that the simplex iterations start at A and that the optimum occurs at H. Indicate whether any of the following paths are *not* legitimate for the simplex algorithm, and state the reason.
     - **(i)** $A \rightarrow B \rightarrow G \rightarrow H$.
     - **(ii)** $A \rightarrow E \rightarrow I \rightarrow H$.
     - **(iii)** $A \rightarrow C \rightarrow E \rightarrow B \rightarrow A \rightarrow D \rightarrow G \rightarrow H$.

4. For the solution space in Figure 3.4, all the constraints are of the type $\leq$ and all the variables x_1, x_2, and x_3 are nonnegative. Suppose that s_1, s_2, s_3, and s_4 (≥ 0) are the slacks associated with constraints represented by the planes $CEIJF$, $BEIHG$, $DFJHG$, and IJH, respectively. Identify the basic and nonbasic variables associated with each feasible extreme point of the solution space.

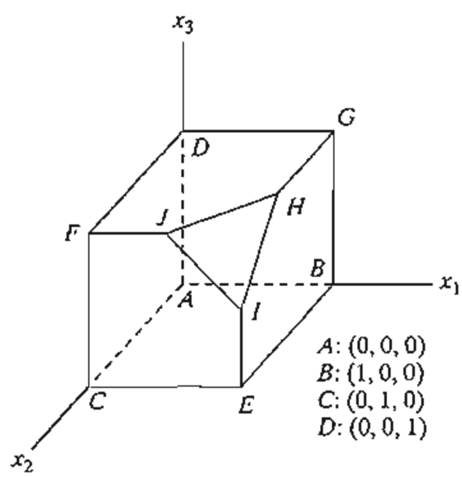


FIGURE 3.4

Solution space of Problem 3, Set 3.2b

5. Consider the solution space in Figure 3.4, where the simplex algorithm starts at point A. Determine the entering variable in the *first* iteration together with its value and the improvement in z for each of the following objective functions:
   - *(a) Maximize $z = x_1 - 2x_2 + 3x_3$
   - (b) Maximize $z = 5x_1 + 2x_2 + 4x_3$
   - (c) Maximize $z = -2x_1 + 7x_2 + 2x_3$
   - (d) Maximize $z = x_1 + x_2 + x_3$

3.3.2 Computational Details of the Simplex Algorithm

This section provides the computational details of a simplex iteration, including the rules for determining the entering and leaving variables as well as for stopping the computations when the optimum solution has been reached. The vehicle of explanation is a numerical example.

Example 3.3-1

We use the Reddy Mikks model (Example 2.1-1) to explain the details of the simplex method. The problem is expressed in equation form as

$$\text{Maximize } z = 5x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

subject to

$$\begin{aligned}
6x_1 + 4x_2 + s_1 \quad\quad\quad\quad &= 24 \quad \text{(Raw material } M1\text{)} \\
x_1 + 2x_2 \quad + s_2 \quad\quad\quad &= 6 \quad \text{(Raw material } M2\text{)} \\
-x_1 + x_2 \quad\quad + s_3 \quad\quad &= 1 \quad \text{(Market limit)} \\
x_2 \quad\quad\quad + s_4 &= 2 \quad \text{(Demand limit)} \\
x_1, x_2, s_1, s_2, s_3, s_4 &\geq 0
\end{aligned}$$

The variables s_1, s_2, s_3, and s_4 are the slacks associated with the respective constraints.

Next, we write the objective equation as

$$z - 5x_1 - 4x_2 = 0$$

In this manner, the starting simplex tableau can be represented as follows:

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	
z	1	−5	−4	0	0	0	0	0	z-row
s_1	0	6	4	1	0	0	0	24	s_1-row
s_2	0	1	2	0	1	0	0	6	s_2-row
s_3	0	−1	1	0	0	1	0	1	s_3-row
s_4	0	0	1	0	0	0	1	2	s_4-row

The design of the tableau specifies the set of basic and nonbasic variables as well as provides the solution associated with the starting iteration. As explained in Section 3.3.1, the simplex iterations start at the origin $(x_1, x_2) = (0, 0)$ whose associated set of nonbasic and basic variables are defined as

$$\text{Nonbasic (zero) variables: } (x_1, x_2)$$

$$\text{Basic variables: } (s_1, s_2, s_3, s_4)$$

Substituting the nonbasic variables $(x_1, x_2) = (0, 0)$ and noting the special 0-1 arrangement of the coefficients of z and the basic variables (s_1, s_2, s_3, s_4) in the tableau, the following solution is immediately available (without any calculations):

$$\begin{aligned} z &= 0 \\ s_1 &= 24 \\ s_2 &= 6 \\ s_3 &= 1 \\ s_4 &= 2 \end{aligned}$$

This information is shown in the tableau by listing the basic variables in the leftmost *Basic* column and their values in the rightmost *Solution* column. In effect, the tableau defines the current corner point by specifying its basic variables and their values, as well as the corresponding value of the objective function, z. Remember that the nonbasic variables (those not listed in the *Basic* column) always equal zero.

Is the starting solution optimal? The objective function $z = 5x_1 + 4x_2$ shows that the solution can be improved by increasing x_1 or x_2. Using the argument in Section 3.3.1, x_1 with the *most positive* coefficient is selected as the *entering variable*. Equivalently, because the simplex tableau expresses the objective function as $z - 5x_1 - 4x_2 = 0$, the entering variable will correspond to the variable with the *most negative* coefficient in the objective equation. This rule is referred to as the **optimality condition**.

The mechanics of determining the leaving variable from the simplex tableau calls for computing the *nonnegative* **ratios** of the right-hand side of the equations (*Solution* column) to the corresponding constraint coefficients under the entering variable, x_1, as the following table shows.

Basic	Entering x_1	Solution	Ratio (or Intercept)
s_1	6	24	$x_1 = \frac{24}{6} = 4 \leftarrow$ minimum
s_2	1	6	$x_1 = \frac{6}{1} = 6$
s_3	-1	1	$x_1 = \frac{1}{-1} = -1$ (ignore)
s_4	0	2	$x_1 = \frac{2}{0} = \infty$ (ignore)
Conclusion: x_1 enters and s_1 leaves			

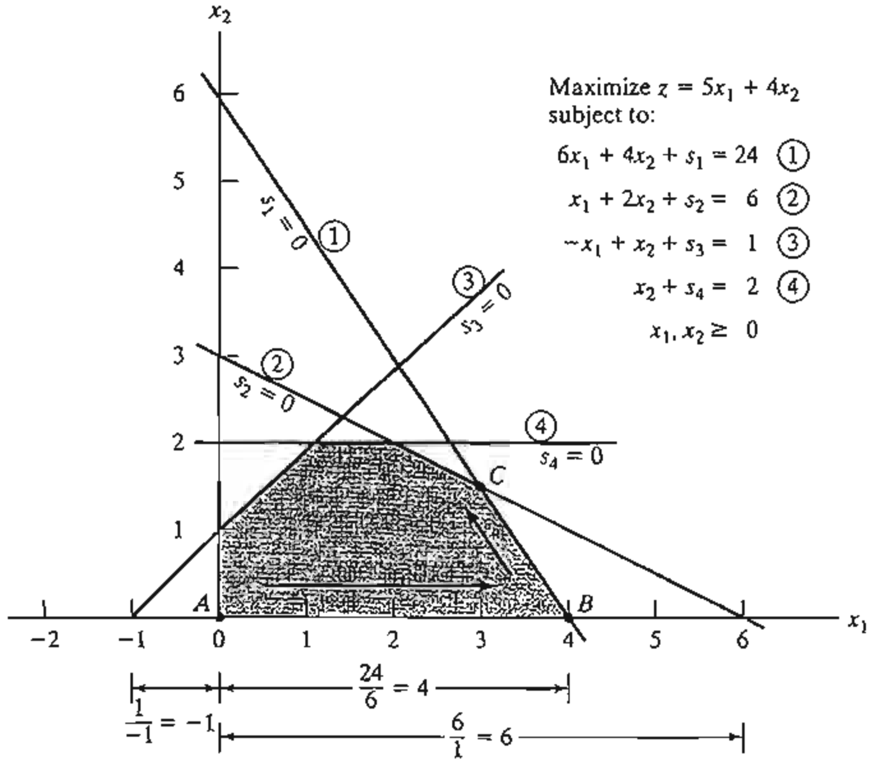


FIGURE 3.5

Graphical interpretation of the simplex method ratios in the Reddy Mikks model

The *minimum nonnegative* ratio automatically identifies the current basic variable s_1 as the leaving variable and assigns the entering variable x_1 the new value of 4.

How do the computed ratios determine the leaving variable and the value of the entering variable? Figure 3.5 shows that the computed ratios are actually the intercepts of the constraints with the entering variable (x_1) axis. We can see that the value of x_1 must be increased to 4 at corner point B, which is the smallest nonnegative intercept with the x_1-axis. An increase beyond B is infeasible. At point B, the current basic variable s_1 associated with constraint 1 assumes a zero value and becomes the *leaving variable*. The rule associated with the ratio computations is referred to as the **feasibility condition** because it guarantees the feasibility of the new solution.

The new solution point B is determined by "swapping" the entering variable x_1 and the leaving variable s_1 in the simplex tableau to produce the following sets of nonbasic and basic variables:

Nonbasic (zero) variables at B: (s_1, x_2)

Basic variables at B: (x_1, s_2, s_3, s_4)

The swapping process is based on the **Gauss-Jordan row operations**. It identifies the entering variable column as the **pivot column** and the leaving variable row as the **pivot row**. The intersection of the pivot column and the pivot row is called the **pivot element**. The following tableau is a restatement of the starting tableau with its pivot row and column highlighted.

	Basic	z	x_1 (Enter ↓)	x_2	s_1	s_2	s_3	s_4	Solution	
	z	1	−5	−4	0	0	0	0	0	
Leave ←	s_1	0	6	4	1	0	0	0	24	Pivot row
	s_2	0	1	2	0	1	0	0	6	
	s_3	0	−1	1	0	0	1	0	1	
	s_4	0	0	1	0	0	0	1	2	
			Pivot column							

The Gauss-Jordan computations needed to produce the new basic solution include two types.

1. *Pivot row*
   a. Replace the leaving variable in the *Basic* column with the entering variable.
   b. New pivot row = Current pivot row ÷ Pivot element
2. *All other rows, including z*
   New Row = (Current row) − (Its pivot column coefficient) ×
   (New pivot row)

These computations are applied to the preceding tableau in the following manner:

1. Replace s_1 in the *Basic* column with x_1:

$$\text{New } x_1\text{-row} = \text{Current } s_1\text{-row} \div 6$$
$$= \tfrac{1}{6}(0\ 6\ 4\ 1\ 0\ 0\ 0\ 24)$$
$$= \left(0\ 1\ \tfrac{2}{3}\ \tfrac{1}{6}\ 0\ 0\ 0\ 4\right)$$

2. New z-row = Current z-row − (−5) × New x_1-row

$$= \left(1\ -5\ -4\ 0\ 0\ 0\ 0\ 0\right) - (-5) \times \left(0\ 1\ \tfrac{2}{3}\ \tfrac{1}{6}\ 0\ 0\ 0\ 4\right)$$
$$= \left(1\ 0\ -\tfrac{2}{3}\ \tfrac{5}{6}\ 0\ 0\ 0\ 20\right)$$

3. New s_2-row = Current s_2-row − (1) × New x_1-row

$$= (0\ 1\ 2\ 0\ 1\ 0\ 0\ 6) - (1) \times \left(0\ 1\ \tfrac{2}{3}\ \tfrac{1}{6}\ 0\ 0\ 0\ 4\right)$$
$$= \left(0\ 0\ \tfrac{4}{3}\ -\tfrac{1}{6}\ 1\ 0\ 0\ 2\right)$$

4. New s_3-row = Current s_3-row − (−1) × New x_1-row

$$= (0\ -1\ 1\ 0\ 0\ 1\ 0\ 1) - (-1) \times \left(0\ 1\ \tfrac{2}{3}\ \tfrac{1}{6}\ 0\ 0\ 0\ 4\right)$$
$$= \left(0\ 0\ \tfrac{5}{3}\ \tfrac{1}{6}\ 0\ 1\ 0\ 5\right)$$

5. New s_4-row = Current s_4-row − (0) × New x_1-row

$$= (0\ 0\ 1\ 0\ 0\ 0\ 1\ 2) - (0)\left(0\ 1\ \tfrac{2}{3}\ \tfrac{1}{6}\ 0\ 0\ 0\ 4\right)$$
$$= (0\ 0\ 1\ 0\ 0\ 0\ 1\ 2)$$

The new basic solution is (x_1, s_2, s_3, s_4), and the new tableau becomes

			↓					
Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	20
x_1	0	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4
← s_2	0	0	$\frac{4}{3}$	$-\frac{1}{6}$	1	0	0	2
s_3	0	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	5
s_4	0	0	1	0	0	0	1	2

Observe that the new tableau has the same properties as the starting tableau. When we set the new nonbasic variables x_2 and s_1 to zero, the *Solution* column automatically yields the new basic solution ($x_1 = 4, s_2 = 2, s_3 = 5, s_4 = 2$). This "conditioning" of the tableau is the result of the application of the Gauss-Jordan row operations. The corresponding new objective value is $z = 20$, which is consistent with

$$\begin{aligned}\text{New } z &= \text{Old } z + \text{New } x_1\text{-value} \times \text{its objective coefficient}\\ &= 0 + 4 \times 5 = 20\end{aligned}$$

In the last tableau, the *optimality condition* shows that x_2 is the entering variable. The feasibility condition produces the following

Basic	Entering x_2	Solution	Ratio
x_1	$\frac{2}{3}$	4	$x_2 = 4 \div \frac{2}{3} = 6$
s_2	$\frac{4}{3}$	2	$x_2 = 2 \div \frac{4}{3} = 1.5$ (minimum)
s_3	$\frac{5}{3}$	5	$x_2 = 5 \div \frac{5}{3} = 3$
s_4	1	2	$x_2 = 2 \div 1 = 2$

Thus, s_2 leaves the basic solution and new value of x_2 is 1.5. The corresponding increase in z is $\frac{2}{3}x_2 = \frac{2}{3} \times 1.5 = 1$, which yields new $z = 20 + 1 = 21$.

Replacing s_2 in the *Basic* column with entering x_2, the following Gauss-Jordan row operations are applied:

1. New pivot x_2-row = Current s_2-row $\div \frac{4}{3}$
2. New z-row = Current z-row $- \left(-\frac{2}{3}\right) \times$ New x_2-row
3. New x_1-row = Current x_1-row $- \left(\frac{2}{3}\right) \times$ New x_2-row
4. New s_3-row = Current s_3-row $- \left(\frac{5}{3}\right) \times$ New x_2-row
5. New s_4-row = Current s_4-row $- (1) \times$ New x_2-row

These computations produce the following tableau:

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x_1	0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
x_2	0	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
s_3	0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
s_4	0	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

Based on the optimality condition, *none* of the z-row coefficients associated with the nonbasic variables, s_1 and s_2, are negative. Hence, the last tableau is optimal.

The optimum solution can be read from the simplex tableau in the following manner. The optimal values of the variables in the *Basic* column are given in the right-hand-side *Solution* column and can be interpreted as

Decision variable	Optimum value	Recommendation
x_1	3	Produce 3 tons of exterior paint daily
x_2	$\frac{3}{2}$	Produce 1.5 tons of interior paint daily
z	21	Daily profit is \$21,000

You can verify that the values $s_1 = s_2 = 0, s_3 = \frac{5}{2}, s_4 = \frac{1}{2}$ are consistent with the given values of x_1 and x_2 by substituting out the values of x_1 and x_2 in the constraints.

The solution also gives the status of the resources. A resource is designated as **scarce** if the activities (variables) of the model use the resource completely. Otherwise, the resource is **abundant**. This information is secured from the optimum tableau by checking the value of the slack variable associated with the constraint representing the resource. If the slack value is zero, the resource is used completely and, hence, is classified as scarce. Otherwise, a positive slack indicates that the resource is abundant. The following table classifies the constraints of the model:

Resource	Slack value	Status
Raw material, $M1$	$s_1 = 0$	Scarce
Raw material, $M2$	$s_2 = 0$	Scarce
Market limit	$s_3 = \frac{5}{2}$	Abundant
Demand limit	$s_4 = \frac{1}{2}$	Abundant

Remarks. The simplex tableau offers a wealth of additional information that includes:

1. *Sensitivity analysis*, which deals with determining the conditions that will keep the current solution unchanged.
2. *Post-optimal analysis*, which deals with finding a new optimal solution when the data of the model are changed.

Section 3.6 deals with sensitivity analysis. The more involved topic of post-optimal analysis is covered in Chapter 4.

TORA Moment.

The Gauss-Jordan computations are tedious, voluminous, and, above all, boring. Yet, they are the least important, because in practice these computations are carried out by the computer. What is important is that you understand *how* the simplex method works. TORA's interactive *user-guided* option (with instant feedback) can be of help in this regard because it allows you to decide the course of the computations in the simplex method without the burden of carrying out the Gauss-Jordan calculations. To use TORA with the Reddy Mikks problem, enter the model and then, from the SOLVE/MODIFY menu, select Solve ⇒ Algebraic ⇒ Iterations ⇒ All-Slack. (The All-Slack selection indicates that the starting basic solution consists of slack variables only. The remaining options will be presented in Sections 3.4, 4.3, and 7.4.2.) Next, click Go To Output Screen. You can generate one or all iterations by clicking Next Iteration or All Iterations. If you opt to generate the iterations one at a time, you can interactively specify the entering and leaving variables by clicking the headings of their corresponding column and row. If your selections are correct, the column turns green and the row turns red. Else, an error message will be posted.

3.3.3 Summary of the Simplex Method

So far we have dealt with the maximization case. In minimization problems, the *optimality condition* calls for selecting the entering variable as the nonbasic variable with the most *positive* objective coefficient in the objective equation, the exact opposite rule of the maximization case. This follows because max z is equivalent to min $(-z)$. As for the *feasibility condition* for selecting the leaving variable, the rule remains unchanged.

Optimality condition. The entering variable in a maximization (minimization) problem is the *nonbasic* variable having the most negative (positive) coefficient in the z-row. Ties are broken arbitrarily. The optimum is reached at the iteration where all the z-row coefficients of the nonbasic variables are nonnegative (nonpositive).

Feasibility condition. For both the maximization and the minimization problems, the leaving variable is the *basic* variable associated with the smallest nonnegative ratio (with *strictly positive* denominator). Ties are broken arbitrarily.

Gauss-Jordan row operations.

1. Pivot row

   a. Replace the leaving variable in the *Basic* column with the entering variable.
   b. New pivot row = Current pivot row ÷ Pivot element

2. All other rows, including z
New row = (Current row) − (pivot column coefficient) × (New pivot row)

The steps of the simplex method are

Step 1. Determine a starting basic feasible solution.

Step 2. Select an *entering variable* using the optimality condition. Stop if there is no entering variable; the last solution is optimal. Else, go to step 3.

Step 3. Select a *leaving variable* using the feasibility condition.

Step 4. Determine the new basic solution by using the appropriate Gauss-Jordan computations. Go to step 2.

PROBLEM SET 3.3B

1. This problem is designed to reinforce your understanding of the simplex feasibility condition. In the first tableau in Example 3.3-1, we used the minimum (nonnegative) ratio test to determine the leaving variable. Such a condition guarantees that none of the new values of the basic variables will become negative (as stipulated by the definition of the LP). To demonstrate this point, force s_2, instead of s_1, to leave the basic solution. Now, look at the resulting simplex tableau, and you will note that s_1 assumes a negative value ($= -12$), meaning that the new solution is infeasible. This situation will never occur if we employ the minimum-ratio feasibility condition.

2. Consider the following set of constraints:

$$x_1 + 2x_2 + 2x_3 + 4x_4 \le 40$$
$$2x_1 - x_2 + x_3 + 2x_4 \le 8$$
$$4x_1 - 2x_2 + x_3 - x_4 \le 10$$
$$x_1, x_2, x_3, x_4 \ge 0$$

Solve the problem for each of the following objective functions.

(a) Maximize $z = 2x_1 + x_2 - 3x_3 + 5x_4$.

(b) Maximize $z = 8x_1 + 6x_2 + 3x_3 - 2x_4$.

(c) Maximize $z = 3x_1 - x_2 + 3x_3 + 4x_4$.

(d) Minimize $z = 5x_1 - 4x_2 + 6x_3 - 8x_4$.

***3.** Consider the following system of equations:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 5x_4 + x_5 \qquad\qquad &= 4 \\ 5x_1 - 2x_2 \qquad + 6x_4 \qquad + x_6 \qquad &= 8 \\ 2x_1 + 3x_2 - 2x_3 + 3x_4 \qquad\qquad + x_7 \quad &= 3 \\ -x_1 \qquad + x_3 - 2x_4 \qquad\qquad\qquad + x_8 &= 0 \end{aligned}$$
$$x_1, x_2, \ldots, x_8 \ge 0$$

Let $x_5, x_6, \ldots,$ and x_8 be a given initial basic feasible solution. Suppose that x_1 becomes basic. Which of the given basic variables must become nonbasic at zero level to guarantee that all the variables remain nonnegative, and what is the value of x_1 in the new solution? Repeat this procedure for x_2, x_3, and x_4.

4. Consider the following LP:

$$\text{Maximize } z = x_1$$

subject to

$$\begin{aligned} 5x_1 + x_2 \qquad\qquad &= 4 \\ 6x_1 \qquad + x_3 \qquad &= 8 \\ 3x_1 \qquad\qquad + x_4 &= 3 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(a) Solve the problem *by inspection* (do not use the Gauss-Jordan row operations), and justify the answer in terms of the basic solutions of the simplex method.

(b) Repeat (a) assuming that the objective function calls for minimizing $z = x_1$.

5. Solve the following problem by *inspection*, and justify the method of solution in terms of the basic solutions of the simplex method.

$$\text{Maximize } z = 5x_1 - 6x_2 + 3x_3 - 5x_4 + 12x_5$$

subject to

$$x_1 + 3x_2 + 5x_3 + 6x_4 + 3x_5 \leq 90$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

(*Hint:* A basic solution consists of one variable only.)

6. The following tableau represents a specific simplex iteration. All variables are nonnegative. The tableau is not optimal for either a maximization or a minimization problem. Thus, when a nonbasic variable enters the solution it can either increase or decrease z or leave it unchanged, depending on the parameters of the entering nonbasic variable.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Solution
z	0	−5	0	4	−1	−10	0	0	620
x_8	0	3	0	−2	−3	−1	5	1	12
x_3	0	1	1	3	1	0	3	0	6
x_1	1	−1	0	0	6	−4	0	0	0

(a) Categorize the variables as basic and nonbasic and provide the current values of all the variables.

*(b) Assuming that the problem is of the maximization type, identify the nonbasic variables that have the potential to improve the value of z. If each such variable enters the basic solution, determine the associated leaving variable, if any, and the associated change in z. Do not use the Gauss-Jordan row operations.

(c) Repeat part (b) assuming that the problem is of the minimization type.

(d) Which nonbasic variable(s) will not cause a change in the value of z when selected to enter the solution?

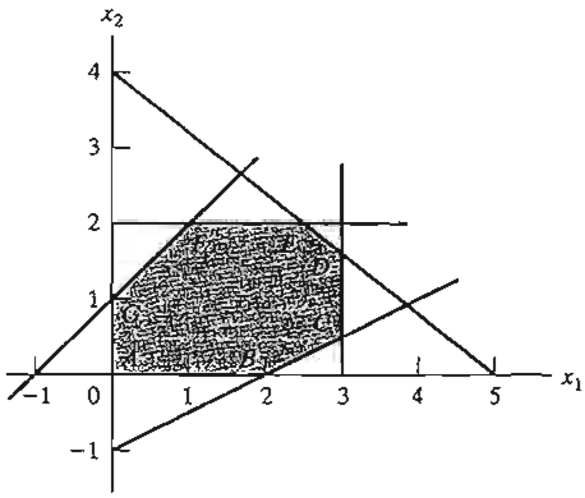


FIGURE 3.6
Solution space for Problem 7, Set 3.3b

7. Consider the two-dimensional solution space in Figure 3.6.
   (a) Suppose that the objective function is given as

$$\text{Maximize } z = 3x_1 + 6x_2$$

   If the simplex iterations start at point A, identify the path to the optimum point E.
   (b) Determine the entering variable, the corresponding ratios of the feasibility condition, and the change in the value of z, assuming that the starting iteration occurs at point A and that the objective function is given as

$$\text{Maximize } z = 4x_1 + x_2$$

   (c) Repeat (b), assuming that the objective function is

$$\text{Maximize } z = x_1 + 4x_2$$

8. Consider the following LP:

$$\text{Maximize } z = 16x_1 + 15x_2$$

subject to

$$40x_1 + 31x_2 \le 124$$
$$-x_1 + x_2 \le 1$$
$$x_1 \le 3$$
$$x_1, x_2 \ge 0$$

   (a) Solve the problem by the simplex method, where the entering variable is the nonbasic variable with the *most* negative z-row coefficient.
   (b) Resolve the problem by the simplex algorithm, always selecting the entering variable as the nonbasic variable with the *least* negative z-row coefficient.
   (c) Compare the number of iterations in (a) and (b). Does the selection of the entering variable as the nonbasic variable with the *most* negative z-row coefficient lead to a smaller number of iterations? What conclusion can be made regarding the optimality condition?
   (d) Suppose that the sense of optimization is changed to minimization by multiplying z by -1. How does this change affect the simplex iterations?

*9. In Example 3.3-1, show how the second best optimal value of z can be determined from the optimal tableau.

10. Can you extend the procedure in Problem 9 to determine the third best optimal value of z?

11. The Gutchi Company manufactures purses, shaving bags, and backpacks. The construction includes leather and synthetics, leather being the scarce raw material. The production process requires two types of skilled labor: sewing and finishing. The following table gives the availability of the resources, their usage by the three products, and the profits per unit.

	Resource requirements per unit			
Resource	*Purse*	*Bag*	*Backpack*	Daily availability
Leather (ft^2)	2	1	3	42 ft^2
Sewing (hr)	2	1	2	40 hr
Finishing (hr)	1	.5	1	45 hr
Selling price (\$)	24	22	45	

(a) Formulate the problem as a linear program and find the optimum solution (using TORA, Excel Solver, or AMPL).

(b) From the optimum solution determine the status of each resource.

12. *TORA experiment.* Consider the following LP:

$$\text{Maximize } z = x_1 + x_2 + 3x_3 + 2x_4$$

subject to

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 5x_4 &\leq 4 \\ 5x_1 - 2x_2 \qquad\quad + 6x_4 &\leq 8 \\ 2x_1 + 3x_2 - 2x_3 + 3x_4 &\leq 3 \\ -x_1 \qquad\quad + x_3 + 2x_4 &\leq 0 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(a) Use TORA's iterations option to determine the optimum tableau.

(b) Select any nonbasic variable to "enter" the basic solution, and click Next Iteration to produce the associated iteration. How does the new objective value compare with the optimum in (a)? The idea is to show that the tableau in (a) is optimum because none of the nonbasic variables can improve the objective value.

13. *TORA experiment.* In Problem 12, use TORA to find the next-best optimal solution.

3.4 ARTIFICIAL STARTING SOLUTION

As demonstrated in Example 3.3-1, LPs in which all the constraints are ($\leq$) with nonnegative right-hand sides offer a convenient all-slack starting basic feasible solution. Models involving ($=$) and/or ($\geq$) constraints do not.

The procedure for starting "ill-behaved" LPs with ($=$) and ($\geq$) constraints is to use **artificial variables** that play the role of slacks at the first iteration, and then dispose of them legitimately at a later iteration. Two closely related methods are introduced here: the M-method and the two-phase method.

3.4.1 *M*-Method

The M-method starts with the LP in equation form (Section 3.1). If equation i does not have a slack (or a variable that can play the role of a slack), an artificial variable, R_i, is added to form a starting solution similar to the convenient all-slack basic solution. However, because the artificial variables are not part of the original LP model, they are assigned a very high **penalty** in the objective function, thus forcing them (eventually) to equal zero in the optimum solution. This will always be the case if the problem has a feasible solution. The following rule shows how the penalty is assigned in the cases of maximization and minimization:

Penalty Rule for Artificial Variables.

Given M, a sufficiently large positive value (mathematically, $M \rightarrow \infty$), the objective coefficient of an artificial variable represents an appropriate **penalty** if:

$$\text{Artificial variable objective coefficient} = \begin{cases} -M, & \text{in maximization problems} \\ M, & \text{in minimization problems} \end{cases}$$

Example 3.4-1

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 &\geq 6 \\ x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Using x_3 as a surplus in the second constraint and x_4 as a slack in the third constraint, the equation form of the problem is given as

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 \qquad\qquad &= 3 \\ 4x_1 + 3x_2 - x_3 \qquad &= 6 \\ x_1 + 2x_2 \qquad + x_4 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The third equation has its slack variable, x_4, but the first and second equations do not. Thus, we add the artificial variables R_1 and R_2 in the first two equations and penalize them in the objective function with $MR_1 + MR_2$ (because we are minimizing). The resulting LP is given as

$$\text{Minimize } z = 4x_1 + x_2 + MR_1 + MR_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 \quad\quad\quad + R_1 \quad\quad &= 3 \\ 4x_1 + 3x_2 - x_3 \quad\quad + R_2 &= 6 \\ x_1 + 2x_2 \quad + x_4 \quad\quad\quad &= 4 \\ x_1, x_2, x_3, x_4, R_1, R_2 &\geq 0 \end{aligned}$$

The associated starting basic solution is now given by $(R_1, R_2, x_4) = (3, 6, 4)$.

From the standpoint of solving the problem on the computer, M must assume a numeric value. Yet, in practically all textbooks, including the first seven editions of this book, M is manipulated algebraically in all the simplex tableaus. The result is an added, and unnecessary, layer of difficulty which can be avoided simply by substituting an appropriate numeric value for M (which is what we do anyway when we use the computer). In this edition, we will break away from the long tradition of manipulating M algebraically and use a numerical substitution instead. The intent, of course, is to simplify the presentation without losing substance.

What value of M should we use? The answer depends on the data of the original LP. Recall that M must be sufficiently large *relative to the original objective coefficients* so it will act as a penalty that forces the artificial variables to zero level in the optimal solution. At the same time, since computers are the main tool for solving LPs, we do not want M to be too large (even though mathematically it should tend to infinity) because potential severe round-off error can result when very large values are manipulated with much smaller values. In the present example, the objective coefficients of x_1 and x_2 are 4 and 1, respectively. It thus appears reasonable to set $M = 100$.

Using $M = 100$, the starting simplex tableau is given as follows (for convenience, the z-column is eliminated because it does not change in all the iterations):

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	−4	−1	0	−100	−100	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	−1	0	1	0	6
x_4	1	2	0	0	0	1	4

Before proceeding with the simplex method computations, we need to make the z-row consistent with the rest of the tableau. Specifically, in the tableau, $x_1 = x_2 = x_3 = 0$, which yields the starting basic solution $R_1 = 3$, $R_2 = 6$, and $x_4 = 4$. This solution yields $z = 100 \times 3 + 100 \times 6 = 900$ (instead of 0, as the right-hand side of the z-row currently shows). This inconsistency stems from the fact that R_1 and R_2 have nonzero coefficients $(-100, -100)$ in the z-row (compare with the all-slack starting solution in Example 3.3-1, where the z-row coefficients of the slacks are zero).

We can eliminate this inconsistency by substituting out R_1 and R_2 in the z-row using the appropriate constraint equations. In particular, notice the highlighted elements $(= 1)$ in the R_1-row and the R_2-row. Multiplying *each* of R_1-row and R_2-row by 100 and adding the *sum* to the z-row will substitute out R_1 and R_2 in the objective row—that is,

$$\text{New } z\text{-row} = \text{Old } z\text{-row} + (100 \times R_1\text{-row} + 100 \times R_2\text{-row})$$

The modified tableau thus becomes (verify!)

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	696	399	−100	0	0	0	900
R_1	3	1	0	1	0	0	3
R_2	4	3	−1	0	1	0	6
x_4	1	2	0	0	0	1	4

Notice that $z = 900$, which is consistent now with the values of the starting basic feasible solution: $R_1 = 3, R_2 = 6$, and $x_4 = 4$.

The last tableau is ready for us to apply the simplex method using the simplex optimality and the feasibility conditions, exactly as we did in Section 3.3.2. Because we are minimizing the objective function, the variable x_1 having the most *positive* coefficient in the z-row (= 696) enters the solution. The minimum ratio of the feasibility condition specifies R_1 as the leaving variable (verify!).

Once the entering and the leaving variables have been determined, the new tableau can be computed by using the familiar Gauss-Jordan operations.

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	0	167	−100	−232	0	0	204
x_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1
R_2	0	$\frac{5}{3}$	−1	$-\frac{4}{3}$	1	0	2
x_4	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3

The last tableau shows that x_2 and R_2 are the entering and leaving variables, respectively. Continuing with the simplex computations, two more iterations are needed to reach the optimum: $x_1 = \frac{2}{5}, x_2 = \frac{9}{5}, z = \frac{17}{5}$ (verify with TORA!).

Note that the artificial variables R_1 and R_2 leave the basic solution in the first and second iterations, a result that is consistent with the concept of penalizing them in the objective function.

Remarks. The use of the penalty M will not force an artificial variable to zero level in the final simplex iteration if the LP does not have a feasible solution (i.e., the constraints are not consistent). In this case, the final simplex iteration will include at least one artificial variable at a positive level. Section 3.5.4 explains this situation.

PROBLEM SET 3.4A

1. Use hand computations to complete the simplex iteration of Example 3.4-1 and obtain the optimum solution.
2. *TORA experiment.* Generate the simplex iterations of Example 3.4-1 using TORA's Iterations ⇒ M-method module (file toraEx3.4-1.txt). Compare the effect of using $M = 1, M = 10$, and $M = 1000$ on the solution. What conclusion can be drawn from this experiment?

3. In Example 3.4-1, identify the starting tableau for each of the following (independent) cases, and develop the associated z-row after substituting out all the artificial variables:
   *(a) The third constraint is $x_1 + 2x_2 \geq 4$.
   *(b) The second constraint is $4x_1 + 3x_2 \leq 6$.
   (c) The second constraint is $4x_1 + 3x_2 = 6$.
   (d) The objective function is to maximize $z = 4x_1 + x_2$.
4. Consider the following set of constraints:

$$\begin{aligned} -2x_1 + 3x_2 &= 3 && (1) \\ 4x_1 + 5x_2 &\geq 10 && (2) \\ x_1 + 2x_2 &\leq 5 && (3) \\ 6x_1 + 7x_2 &\leq 3 && (4) \\ 4x_1 + 8x_2 &\geq 5 && (5) \\ x_1, x_2 &\geq 0 \end{aligned}$$

   For each of the following problems, develop the z-row after substituting out the artificial variables:
   (a) Maximize $z = 5x_1 + 6x_2$ subject to (1), (3), and (4).
   (b) Maximize $z = 2x_1 - 7x_2$ subject to (1), (2), (4), and (5).
   (c) Minimize $z = 3x_1 + 6x_2$ subject to (3), (4), and (5).
   (d) Minimize $z = 4x_1 + 6x_2$ subject to (1), (2), and (5).
   (e) Minimize $z = 3x_1 + 2x_2$ subject to (1) and (5).
5. Consider the following set of constraints:

$$\begin{aligned} x_1 + x_2 + x_3 &= 7 \\ 2x_1 - 5x_2 + x_3 &\geq 10 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

   Solve the problem for each of the following objective functions:
   (a) Maximize $z = 2x_1 + 3x_2 - 5x_3$.
   (b) Minimize $z = 2x_1 + 3x_2 - 5x_3$.
   (c) Maximize $z = x_1 + 2x_2 + x_3$.
   (d) Minimize $z = 4x_1 - 8x_2 + 3x_3$.

*6. Consider the problem

$$\text{Maximize } z = 2x_1 + 4x_2 + 4x_3 - 3x_4$$

   subject to

$$\begin{aligned} x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 4x_2 \quad + x_4 &= 8 \\ x_1, x_2, x_3, x_4, &\geq 0 \end{aligned}$$

   The problem shows that x_3 and x_4 can play the role of slacks for the two equations. They differ from slacks in that they have nonzero coefficients in the objective function. We can use x_3 and x_4 as starting variable, but, as in the case of artificial variables, they must be substituted out in the objective function before the simplex iterations are carried out. Solve the problem with x_3 and x_4 as the starting basic variables and without using any artificial variables.

7. Solve the following problem using x_3 and x_4 as starting basic feasible variables. As in Problem 6, do not use any artificial variables.

$$\text{Minimize } z = 3x_1 + 2x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 + 4x_2 + x_3 \qquad &\geq 7 \\ 2x_1 + x_2 \qquad + x_4 &\geq 10 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

8. Consider the problem

$$\text{Maximize } z = x_1 + 5x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 - x_2 \qquad &= 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The variable x_3 plays the role of a slack. Thus, no artificial variable is needed in the first constraint. However, in the second constraint, an artificial variable is needed. Use this starting solution (i.e., x_3 in the first constraint and R_2 in the second constraint) to solve this problem.

9. Show how the M-method will indicate that the following problem has no feasible solution.

$$\text{Maximize } z = 2x_1 + 5x_2$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 &\geq 6 \\ 2x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

3.4.2 Two-Phase Method

In the M-method, the use of the penalty M, which by definition must be large relative to the actual objective coefficients of the model, can result in roundoff error that may impair the accuracy of the simplex calculations. The two-phase method alleviates this difficulty by eliminating the constant M altogether. As the name suggests, the method solves the LP in two phases: Phase I attempts to find a starting basic feasible solution, and, if one is found, Phase II is invoked to solve the original problem.

Summary of the Two-Phase Method

Phase I. Put the problem in equation form, and add the necessary artificial variables to the constraints (exactly as in the M-method) to secure a starting basic solution. Next, find a basic solution of the resulting equations that, regardless of whether the LP is maximization or minimization, *always* minimizes the sum of the artificial variables. If the minimum value of the sum is positive, the LP problem has no feasible solution, which ends the process (recall that a positive artificial variable signifies that an original constraint is not satisfied). Otherwise, proceed to Phase II.

Phase II. Use the feasible solution from Phase I as a starting basic feasible solution for the *original* problem.

Example 3.4-2

We use the same problem in Example 3.4-1.

Phase I

$$\text{Minimize } r = R_1 + R_2$$

subject to

$$3x_1 + x_2 + R_1 = 3$$
$$4x_1 + 3x_2 - x_3 + R_2 = 6$$
$$x_1 + 2x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4, R_1, R_2 \geq 0$$

The associated tableau is given as

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	−1	−1	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	−1	0	1	0	6
x_4	1	2	0	0	0	1	4

As in the M-method, R_1 and R_2 are substituted out in the r-row by using the following computations:

$$\text{New } r\text{-row} = \text{Old } r\text{-row} + (1 \times R_1\text{-row} + 1 \times R_2\text{-row})$$

The new r-row is used to solve Phase I of the problem, which yields the following optimum tableau (verify with TORA's Iterations ⇒ Two-phase Method):

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	−1	−1	0	0
x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	−1	1	1

Because minimum $r = 0$, Phase I produces the basic feasible solution $x_1 = \frac{3}{5}$, $x_2 = \frac{6}{5}$, and $x_4 = 1$. At this point, the artificial variables have completed their mission, and we can eliminate their columns altogether from the tableau and move on to Phase II.

Phase II

After deleting the artificial columns, we write the *original* problem as

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$\begin{aligned} x_1 \quad + \tfrac{1}{5}x_3 \quad &= \tfrac{3}{5} \\ x_2 - \tfrac{3}{5}x_3 \quad &= \tfrac{6}{5} \\ x_3 + x_4 &= 1 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Essentially, Phase I is a procedure that transforms the original constraint equations in a manner that provides a starting basic feasible solution for the problem, if one exists. The tableau associated with Phase II problem is thus given as

Basic	x_1	x_2	x_3	x_4	Solution
z	-4	-1	0	0	0
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	1

Again, because the basic variables x_1 and x_2 have nonzero coefficients in the z-row, they must be substituted out, using the following computations.

$$\text{New } z\text{-row} = \text{Old } z\text{-row} + (4 \times x_1\text{-row} + 1 \times x_2\text{-row})$$

The initial tableau of Phase II is thus given as

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	$\frac{1}{5}$	0	$\frac{18}{5}$
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	1

Because we are minimizing, x_3 must enter the solution. Application of the simplex method will produce the optimum in one iteration (verify with TORA).

Remarks. Practically all commercial packages use the two-phase method to solve LP. The *M*-method with its potential adverse roundoff error is probably never used in practice. Its inclusion in this text is purely for historical reasons, because its development predates the development of the two-phase method.

The removal of the artificial variables and their columns at the end of Phase I can take place only when they are all *nonbasic* (as Example 3.4-2 illustrates). If one or more artificial variables are *basic* (at *zero* level) at the end of Phase I, then the following additional steps must be undertaken to remove them prior to the start of Phase II.

Step 1. Select a zero artificial variable to leave the basic solution and designate its row as the *pivot row*. The entering variable can be *any* nonbasic (nonartificial) variable with a *nonzero* (positive or negative) coefficient in the pivot row. Perform the associated simplex iteration.

Step 2. Remove the column of the (just-leaving) artificial variable from the tableau. If all the zero artificial variables have been removed, go to Phase II. Otherwise, go back to Step 1.

The logic behind Step 1 is that the feasibility of the remaining basic variables will not be affected when a zero artificial variable is made nonbasic regardless of whether the pivot element is positive or negative. Problems 5 and 6, Set 3.4b illustrate this situation. Problem 7 provides an additional detail about Phase I calculations.

PROBLEM SET 3.4B

***1.** In Phase I, if the LP is of the maximization type, explain why we do not maximize the sum of the artificial variables in Phase I.

2. For each case in Problem 4, Set 3.4a, write the corresponding Phase I objective function.

3. Solve Problem 5, Set 3.4a, by the two-phase method.

4. Write Phase I for the following problem, and then solve (with TORA for convenience) to show that the problem has no feasible solution.

$$\text{Maximize } z = 2x_1 + 5x_2$$

subject to

$$3x_1 + 2x_2 \geq 6$$
$$2x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

5. Consider the following problem:

$$\text{Maximize } z = 2x_1 + 2x_2 + 4x_3$$

subject to

$$2x_1 + x_2 + x_3 \leq 2$$
$$3x_1 + 4x_2 + 2x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0$$

(a) Show that Phase I will terminate with an artificial *basic* variable at zero level (you may use TORA for convenience).

(b) Remove the zero artificial variable prior to the start of Phase II, then carry out Phase II iterations.

6. Consider the following problem:

$$\text{Maximize } z = 3x_1 + 2x_2 + 3x_3$$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 2 \\ x_1 + 3x_2 + x_3 &= 6 \\ 3x_1 + 4x_2 + 2x_3 &= 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(a) Show that Phase I terminates with two zero artificial variables in the basic solution (use TORA for convenience).

(b) Show that when the procedure of Problem 5(b) is applied at the end of Phase I, only one of the two zero artificial variables can be made nonbasic.

(c) Show that the original constraint associated with the zero artificial variable that cannot be made nonbasic in (b) must be redundant—hence, its row and its column can be dropped altogether at the start of Phase II.

*7. Consider the following LP:

$$\text{Maximize } z = 3x_1 + 2x_2 + 3x_3$$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 &\leq 2 \\ 3x_1 + 4x_2 + 2x_3 &\geq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The optimal simplex tableau at the end of Phase I is given as

Basic	x_1	x_2	x_3	x_4	x_5	R	Solution
z	-5	0	-2	-1	-4	0	0
x_2	2	1	1	0	1	0	2
R	-5	0	-2	-1	-4	1	0

Explain why the nonbasic variables x_1, x_3, x_4, and x_5 can never assume positive values at the end of Phase II. Hence, conclude that their columns can dropped before we start Phase II. In essence, the removal of these variables reduces the constraint equations of the problem to $x_2 = 2$. This means that it will not be necessary to carry out Phase II at all, because the solution space is reduced to one point only.

8. Consider the LP model

$$\text{Minimize } z = 2x_1 - 4x_2 + 3x_3$$

subject to

$$\begin{aligned} 5x_1 - 6x_2 + 2x_3 &\geq 5 \\ -x_1 + 3x_2 + 5x_3 &\geq 8 \\ 2x_1 + 5x_2 - 4x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Show how the inequalities can be modified to a set of equations that requires the use of a single artificial variable only (instead of two).

3.5 SPECIAL CASES IN THE SIMPLEX METHOD

This section considers four special cases that arise in the use of the simplex method.

1. Degeneracy
2. Alternative optima
3. Unbounded solutions
4. Nonexisting (or infeasible) solutions

Our interest in studying these special cases is twofold: (1) to present a *theoretical* explanation of these situations and (2) to provide a *practical* interpretation of what these special results could mean in a real-life problem.

3.5.1 Degeneracy

In the application of the feasibility condition of the simplex method, a tie for the minimum ratio may occur and can be broken arbitrarily. When this happens, at least one *basic* variable will be zero in the next iteration and the new solution is said to be **degenerate.**

There is nothing alarming about a degenerate solution, with the exception of a small theoretical inconvenience, called **cycling** or **circling**, which we shall discuss shortly. From the practical standpoint, the condition reveals that the model has at least one *redundant* constraint. To provide more insight into the practical and theoretical impacts of degeneracy, a numeric example is used.

Example 3.5-1 (Degenerate Optimal Solution)

$$\text{Maximize } z = 3x_1 + 9x_2$$

subject to

$$x_1 + 4x_2 \leq 8$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Given the slack variables x_3 and x_4, the following tableaus provide the simplex iterations of the problem:

Iteration	Basic	x_1	x_2	x_3	x_4	Solution
0	z	-3	-9	0	0	0
x_2 enters	x_3	1	4	1	0	8
x_3 leaves	x_4	1	2	0	1	4
1	z	$-\frac{3}{4}$	0	$\frac{9}{4}$	0	18
x_1 enters	x_2	$\frac{1}{4}$	1	$\frac{1}{4}$	0	2
x_4 leaves	x_4	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0
2	z	0	0	$\frac{3}{2}$	$\frac{3}{2}$	18
(optimum)	x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	2
	x_1	1	0	-1	2	0

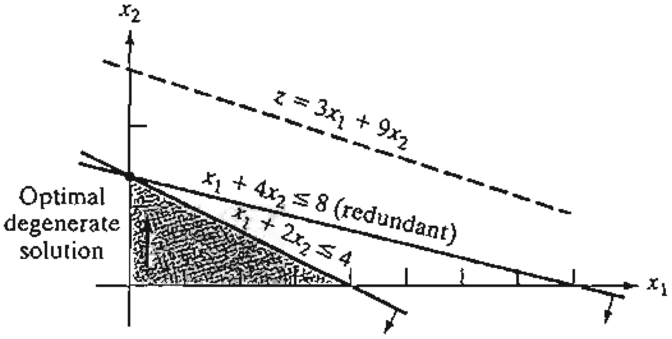


FIGURE 3.7

LP degeneracy in Example 3.5-1

In iteration 0, x_3 and x_4 tie for the leaving variable, leading to degeneracy in iteration 1 because the basic variable x_4 assumes a zero value. The optimum is reached in one additional iteration.

What is the practical implication of degeneracy? Look at the graphical solution in Figure 3.7. Three lines pass through the optimum point ($x_1 = 0, x_2 = 2$). Because this is a two-dimensional problem, the point is *overdetermined* and one of the constraints is redundant.[2] In practice, the mere knowledge that some resources are superfluous can be valuable during the implementation of the solution. The information may also lead to discovering irregularities in the construction of the model. Unfortunately, there are no efficient computational techniques for identifying the redundant constraints directly from the tableau.

From the theoretical standpoint, degeneracy has two implications. The first is the phenomenon of **cycling** or **circling**. Looking at simplex iterations 1 and 2, you will notice that the objective value does not improve ($z = 18$). It is thus possible for the simplex method to enter a repetitive sequence of iterations, never improving the objective value and never satisfying the optimality condition (see Problem 4, Set 3.5a). Although there are methods for eliminating cycling, these methods lead to drastic slowdown in computations. For this reason, most LP codes do not include provisions for cycling, relying on the fact that it is a rare occurrence in practice.

The second theoretical point arises in the examination of iterations 1 and 2. Both iterations, though differing in the basic-nonbasic categorization of the variables, yield identical values for all the variables and objective value—namely,

$$x_1 = 0, x_2 = 2, x_3 = 0, x_4 = 0, z = 18$$

Is it possible then to stop the computations at iteration 1 (when degeneracy first appears), even though it is not optimum? The answer is no, because the solution may be *temporarily* degenerate as Problem 2, Set 3.5a demonstrates.

[2]Redundancy generally implies that constraints can be removed without affecting the feasible solution space. A sometimes quoted counterexample is $x + y \leq 1, x \geq 1, y \geq 0$. Here, the removal of any one constraint will change the feasible space from a single point to a region. Suffice it to say, however, that this condition is true only if the solution space consists of a single feasible point, a highly unlikely occurrence in real-life LPs.

PROBLEM SET 3.5A

*1. Consider the graphical solution space in Figure 3.8. Suppose that the simplex iterations start at A and that the optimum solution occurs at D. Further, assume that the objective function is defined such that at A, x_1 enters the solution first.

   (a) Identify (on the graph) the corner points that define the simplex method path to the optimum point.

   (b) Determine the maximum possible number of simplex iterations needed to reach the optimum solution, assuming no cycling.

2. Consider the following LP:

$$\text{Maximize } z = 3x_1 + 2x_2$$

subject to

$$4x_1 - x_2 \le 8$$
$$4x_1 + 3x_2 \le 12$$
$$4x_1 + x_2 \le 8$$
$$x_1, x_2 \ge 0$$

   (a) Show that the associated simplex iterations are temporarily degenerate (you may use TORA for convenience).

   (b) Verify the result by solving the problem graphically (TORA's Graphic module can be used here).

3. *TORA experiment.* Consider the LP in Problem 2.

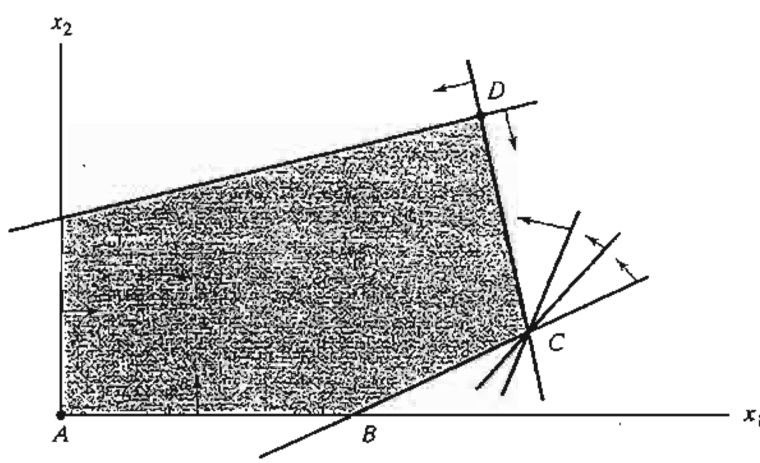
   (a) Use TORA to generate the simplex iterations. How many iterations are needed to reach the optimum?

   (b) Interchange constraints (1) and (3) and re-solve the problem with TORA. How many iterations are needed to solve the problem?

   (c) Explain why the numbers of iterations in (a) and (b) are different.

FIGURE 3.8

Solution space of Problem 1, Set 3.5a



4. *TORA Experiment.* Consider the following LP (authored by E.M. Beale to demonstrate cycling):

$$\text{Maximize } z = \tfrac{3}{4}x_1 - 20x_2 + \tfrac{1}{2}x_3 - 6x_4$$

subject to

$$\tfrac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 \le 0$$
$$\tfrac{1}{2}x_1 - 12x_2 - \tfrac{1}{2}x_3 + 3x_4 \le 0$$
$$x_3 \le 1$$
$$x_1, x_2, x_3, x_4 \ge 0$$

From TORA's SOLVE/MODIFY menu, select Solve ⇒ Algebraic ⇒ Iterations ⇒ All-slack. Next, "thumb" through the successive simplex iterations using the command Next iteration (do not use All iterations, because the simplex method will then cycle indefinitely). You will notice that the starting all-slack basic feasible solution at iteration 0 will reappear identically in iteration 6. This example illustrates the occurrence of cycling in the simplex iterations and the possibility that the algorithm may never converge to the optimum solution.

It is interesting that cycling will not occur in this example if all the coefficients in this LP are converted to integer values by using proper multiples (try it!).

3.5.2 Alternative Optima

When the objective function is parallel to a nonredundant **binding constraint** (i.e., a constraint that is satisfied as an equation at the optimal solution), the objective function can assume the same optimal value at more than one solution point, thus giving rise to alternative optima. The next example shows that there is an *infinite* number of such solutions. It also demonstrates the practical significance of encountering such solutions.

Example 3.5-2 (Infinite Number of Solutions)

$$\text{Maximize } z = 2x_1 + 4x_2$$

subject to

$$x_1 + 2x_2 \le 5$$
$$x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$

Figure 3.9 demonstrates how alternative optima can arise in the LP model when the objective function is parallel to a binding constraint. Any point on the *line segment BC* represents an alternative optimum with the same objective value $z = 10$.

The iterations of the model are given by the following tableaus.

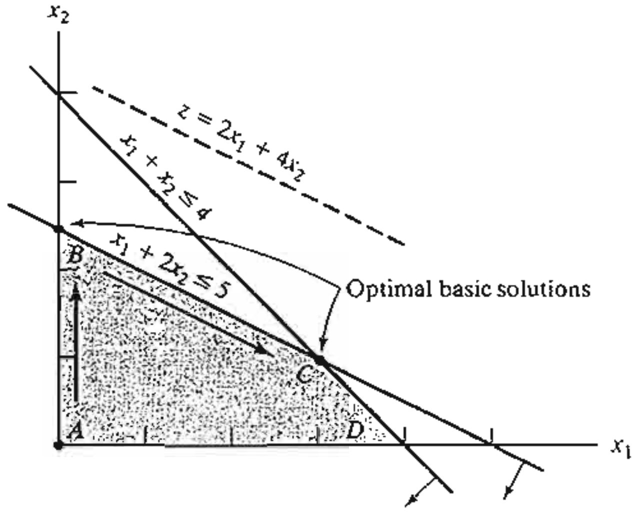


FIGURE 3.9
LP alternative optima in Example 3.5-2

Iteration	Basic	x_1	x_2	x_3	x_4	Solution
0	z	−2	−4	0	0	0
x_2 enters	x_3	1	2	1	0	5
x_3 leaves	x_4	1	1	0	1	4
1 (optimum)	z	0	0	2	0	10
x_1 enters	x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{5}{2}$
x_4 leaves	x_4	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	$\frac{3}{2}$
2	z	0	0	2	0	10
(alternative optimum)	x_2	0	1	1	−1	1
	x_1	1	0	−1	2	3

Iteration 1 gives the optimum solution $x_1 = 0, x_2 = \frac{5}{2}$, and $z = 10$, which coincides with point B in Figure 3.9. How do we know from this tableau that alternative optima exist? Look at the z-equation coefficients of the *non*basic variables in iteration 1. The coefficient of nonbasic x_1 is zero, indicating that x_1 can enter the basic solution without changing the value of z, but causing a change in the values of the variables. Iteration 2 does just that—letting x_1 enter the basic solution and forcing x_4 to leave. The new solution point occurs at $C(x_1 = 3, x_2 = 1, z = 10)$. (TORA's Iterations option allows determining one alternative optimum at a time.)

The simplex method determines only the two corner points B and C. Mathematically, we can determine all the points (x_1, x_2) on the line segment BC as a nonnegative weighted average of points B and C. Thus, given

$$B: x_1 = 0, x_2 = \frac{5}{2}$$

$$C: x_1 = 3, x_2 = 1$$

then all the points on the line segment BC are given by

$$\left.\begin{aligned}\hat{x}_1 &= \alpha(0) + (1-\alpha)(3) = 3 - 3\alpha \\ \hat{x}_2 &= \alpha\left(\tfrac{5}{2}\right) + (1-\alpha)(1) = 1 + \tfrac{3}{2}\alpha\end{aligned}\right\}, 0 \le \alpha \le 1$$

When $\alpha = 0$, $(\hat{x}_1, \hat{x}_2) = (3,1)$, which is point C. When $\alpha = 1$, $(\hat{x}_1, \hat{x}_2) = \left(0, \frac{5}{2}\right)$, which is point B. For values of α between 0 and 1, $(\hat{x}_1, \hat{x}_2)$ lies between B and C.

Remarks. In practice, alternative optima are useful because we can choose from many solutions without experiencing deterioration in the objective value. For instance, in the present example, the solution at B shows that activity 2 only is at a positive level, whereas at C both activities are positive. If the example represents a product-mix situation, there may be advantages in producing two products rather than one to meet market competition. In this case, the solution at C may be more appealing.

PROBLEM SET 3.5B

*1. For the following LP, identify three alternative optimal basic solutions, and then write a general expression for all the nonbasic alternative optima comprising these three basic solutions.

$$\text{Maximize } z = x_1 + 2x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\le 10 \\ x_1 + x_2 \quad\quad &\le 5 \\ x_1 \quad\quad\quad\quad &\le 1 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Note: Although the problem has more than three alternative basic solution optima, you are only required to identify three of them. You may use TORA for convenience.

2. Solve the following LP:

$$\text{Maximize } z = 2x_1 - x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 - x_2 + 5x_3 &\le 10 \\ 2x_1 - x_2 + 3x_3 &\le 40 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

From the optimal tableau, show that all the alternative optima are not corner points (i.e., nonbasic). Give a two-dimensional graphical demonstration of the type of solution space and objective function that will produce this result. (You may use TORA for convenience.)

3. For the following LP, show that the optimal solution is degenerate and that none of the alternative solutions are corner points (you may use TORA for convenience).

$$\text{Maximize } z = 3x_1 + x_2$$

subject to

$$x_1 + 2x_2 \leq 5$$
$$x_1 + x_2 - x_3 \leq 2$$
$$7x_1 + 3x_2 - 5x_3 \leq 20$$
$$x_1, x_2, x_3 \geq 0$$

3.5.3 Unbounded Solution

In some LP models, the values of the variables may be increased indefinitely without violating any of the constraints—meaning that the solution space is *unbounded* in at least one variable. As a result, the objective value may increase (maximization case) or decrease (minimization case) indefinitely. In this case, both the solution space and the optimum objective value are unbounded.

Unboundedness points to the possibility that the model is poorly constructed. The most likely irregularity in such models is that one or more nonredundant constraints have not been accounted for, and the parameters (constants) of some constraints may not have been estimated correctly.

The following examples show how unboundedness, in both the solution space and the objective value, can be recognized in the simplex tableau.

Example 3.5-3 (Unbounded Objective Value)

$$\text{Maximize } z = 2x_1 + x_2$$

subject to

$$x_1 - x_2 \leq 10$$
$$2x_1 \leq 40$$
$$x_1, x_2 \geq 0$$

Starting Iteration

Basic	x_1	x_2	x_3	x_4	Solution
z	-2	-1	0	0	0
x_3	1	-1	1	0	10
x_4	2	0	0	1	40

In the starting tableau, both x_1 and x_2 have negative z-equation coefficients. Hence either one can improve the solution. Because x_1 has the most negative coefficient, it is normally selected as the entering variable. However, *all* the *constraint* coefficients under x_2 (i.e., the denominators of the ratios of the feasibility condition) are *negative* or *zero*. This means that there is no leaving variable and that x_2 can be increased indefinitely without violating any of the constraints (compare with the graphical interpretation of the minimum ratio in Figure 3.5). Because each unit increase in x_2 will increase z by 1, an infinite increase in x_2 leads to an infinite increase in z. Thus, the problem has no bounded solution. This result can be seen in Figure 3.10. The solution space is unbounded in the direction of x_2, and the value of z can be increased indefinitely.

Remarks. What would have happened if we had applied the strict optimality condition that calls for x_1 to enter the solution? The answer is that a succeeding tableau would eventually have led to an entering variable with the same characteristics as x_2. See Problem 1, Set3.5c.

PROBLEM SET 3.5C

1. *TORA Experiment*. Solve Example 3.5-3 using TORA's Iterations option and show that even though the solution starts with x_1 as the entering variable (per the optimality condition), the simplex algorithm will point eventually to an unbounded solution.

*2. Consider the LP:

$$\text{Maximize } z = 20x_1 + 10x_2 + x_3$$

subject to

$$\begin{aligned} 3x_1 - 3x_2 + 5x_3 &\leq 50 \\ x_1 \qquad + x_3 &\leq 10 \\ x_1 - x_2 + 4x_3 &\leq 20 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

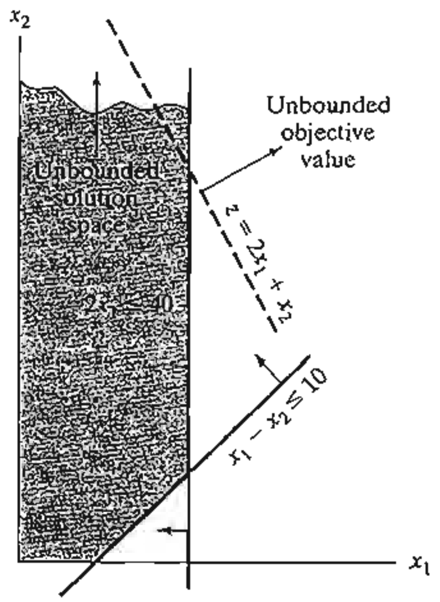


FIGURE 3.10
LP unbounded solution in Example 3.5-3

(a) By inspecting the constraints, determine the direction (x_1, x_2, or x_3) in which the solution space is unbounded.

(b) Without further computations, what can you conclude regarding the optimum objective value?

3. In some ill-constructed LP models, the solution space may be unbounded even though the problem may have a bounded objective value. Such an occurrence can point only to irregularities in the construction of the model. In large problems, it may be difficult to detect unboundedness by inspection. Devise a procedure for determining whether or not a solution space is unbounded.

3.5.4 Infeasible Solution

LP models with inconsistent constraints have no feasible solution. This situation can never occur if *all* the constraints are of the type $\leq$ with nonnegative right-hand sides because the slacks provide a feasible solution. For other types of constraints, we use artificial variables. Although the artificial variables are penalized in the objective function to force them to zero at the optimum, this can occur only if the model has a feasible space. Otherwise, at least one artificial variable will be *positive* in the optimum iteration. From the practical standpoint, an infeasible space points to the possibility that the model is not formulated correctly.

Example 3.5-4 (Infeasible Solution Space)

Consider the following LP:

$$\text{Maximize } z = 3x_1 + 2x_2$$

subject to

$$2x_1 + x_2 \leq 2$$

$$3x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

Using the penalty $M = 100$ for the artificial variable R, the following tableaux provide the simplex iterations of the model.

Iteration	Basic	x_1	x_2	x_4	x_3	R	Solution
0	z	−303	−402	100	0	0	−1200
x_2 enters	x_3	2	1	0	1	0	2
x_3 leaves	R	3	4	−1	0	1	12
1	z	501	0	100	402	0	−396
(pseudo-optimum)	x_2	2	1	0	1	0	2
	R	−5	0	−1	−4	1	4

Optimum iteration 1 shows that the artificial variable R is *positive* (= 4), which indicates that the problem is infeasible. Figure 3.11 demonstrates the infeasible solution space. By allowing

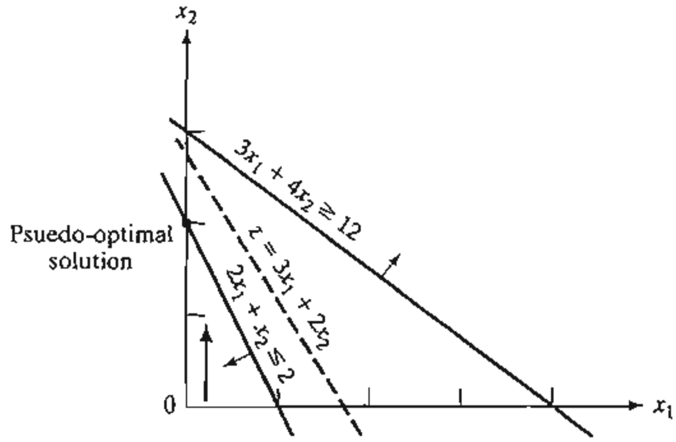


FIGURE 3.11
Infeasible solution of Example 3.5-4

the artificial variable to be positive, the simplex method, in essence, has reversed the direction of the inequality from $3x_1 + 4x_2 \geq 12$ to $3x_1 + 4x_2 \leq 12$ (can you explain how?). The result is what we may call a **pseudo-optimal** solution.

PROBLEM SET 3.5D

*1. Toolco produces three types of tools, *T1*, *T2*, and *T3*. The tools use two raw materials, *M1* and *M2*, according to the data in the following table:

	Number of units of raw materials per tool		
Raw material	*T1*	*T2*	*T3*
M1	3	5	6
M2	5	3	4

The available daily quantities of raw materials *M1* and *M2* are 1000 units and 1200 units, respectively. The marketing department informed the production manager that according to their research, the daily demand for all three tools must be at least 500 units. Will the manufacturing department be able to satisfy the demand? If not, what is the most Toolco can provide of the three tools?

2. *TORA Experiment.* Consider the LP model

$$\text{Maximize } z = 3x_1 + 2x_1 + 3x_3$$

subject to

$$2x_1 + x_2 + x_3 \leq 2$$
$$3x_1 + 4x_2 + 2x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0$$

Use TORA's Iterations ⇒ M-Method to show that the optimal solution includes an artificial basic variable, but at zero level. Does the problem have a *feasible* optimal solution?

3.6 SENSITIVITY ANALYSIS

In LP, the parameters (input data) of the model can change within certain limits without causing the optimum solution to change. This is referred to as *sensitivity analysis*, and will be the subject matter of this section. Later, in Chapter 4, we will study *post-optimal analysis* which deals with determining the new optimum solution resulting from making targeted changes in the input data.

In LP models, the parameters are usually not exact. With sensitivity analysis, we can ascertain the impact of this uncertainty on the quality of the optimum solution. For example, for an estimated unit profit of a product, if sensitivity analysis reveals that the optimum remains the same for a $\pm 10\%$ change in the unit profit, we can conclude that the solution is more robust than in the case where the indifference range is only $\pm 1\%$.

We will start with the more concrete graphical solution to explain the basics of sensitivity analysis. These basics will then be extended to the general LP problem using the simplex tableau results.

3.6.1 Graphical Sensitivity Analysis

This section demonstrates the general idea of sensitivity analysis. Two cases will be considered:

1. Sensitivity of the optimum solution to changes in the availability of the resources (right-hand side of the constraints).
2. Sensitivity of the optimum solution to changes in unit profit or unit cost (coefficients of the objective function).

We will consider the two cases separately, using examples of two-variable graphical LPs.

Example 3.6-1 (Changes in the Right-Hand Side)

JOBCO produces two products on two machines. A unit of product 1 requires 2 hours on machine 1 and 1 hour on machine 2. For product 2, a unit requires 1 hour on machine 1 and 3 hours on machine 2. The revenues per unit of products 1 and 2 are \$30 and \$20, respectively. The total daily processing time available for each machine is 8 hours.

Letting x_1 and x_2 represent the daily number of units of products 1 and 2, respectively, the LP model is given as

$$\text{Maximize } z = 30x_1 + 20x_2$$

subject to

$$2x_1 + x_2 \le 8 \quad \text{(Machine 1)}$$

$$x_1 + 3x_2 \le 8 \quad \text{(Machine 2)}$$

$$x_1, x_2 \ge 0$$

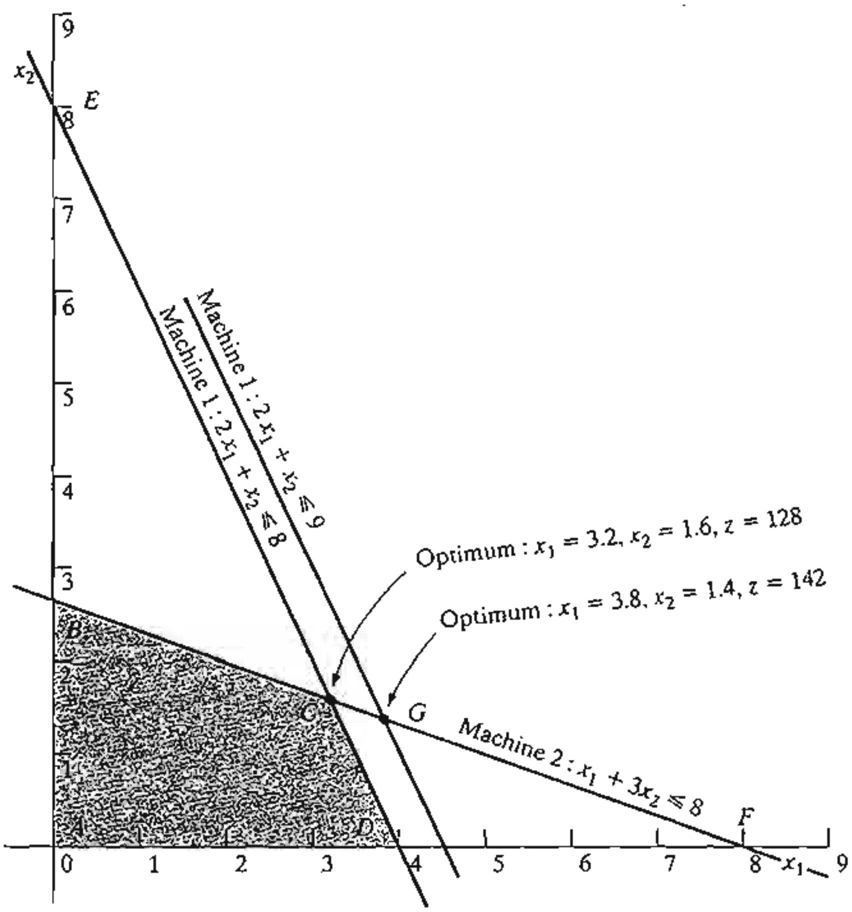
Figure 3.12 illustrates the change in the optimum solution when changes are made in the capacity of machine 1. If the daily capacity is increased from 8 hours to 9 hours, the new optimum will occur at point G. The rate of change in optimum z resulting from changing machine 1 capacity from 8 hours to 9 hours can be computed as follows:

$$\left(\begin{array}{c}\text{Rate of revenue change}\\ \text{resulting from increasing}\\ \text{machine 1 capacity by 1 hr}\\ \text{(point } C \text{ to point } G\text{)}\end{array}\right) = \frac{z_G - z_C}{\text{(Capacity change)}} = \frac{142 - 128}{9 - 8} = \$14.00/\text{hr}$$

The computed rate provides a *direct link* between the model input (resources) and its output (total revenue) that represents the **unit worth of a resource** (in \$/hr)—that is, the change in the optimal objective value per unit change in the availability of the resource (machine capacity). This means that a unit increase (decrease) in machine 1 capacity will increase (decrease) revenue by \$14.00. Although *unit worth of a resource* is an apt description of the rate of change of the objective function, the technical name **dual** or **shadow price** is now standard in the LP literature and all software packages and, hence, will be used throughout the book.

FIGURE 3.12

Graphical sensitivity of optimal solution to changes in the availability of resources (right-hand side of the constraints)



Looking at Figure 3.12, we can see that the dual price of \$14.00/hr remains valid for changes (increases or decreases) in machine 1 capacity that move its constraint parallel to itself to any point on the line segment BF. This means that the range of applicability of the given dual price can be computed as follows:

$$\text{Minimum machine 1 capacity [at } B = (0, 2.67)] = 2 \times 0 + 1 \times 2.67 = 2.67 \text{ hr}$$

$$\text{Maximum machine 1 capacity [at } F = (8, 0)] = 2 \times 8 + 1 \times 0 = 16 \text{ hr}$$

We can thus conclude that the dual price of \$14.00/hr will remain valid for the range

$$2.67 \text{ hrs} \leq \text{Machine 1 capacity} \leq 16 \text{ hrs}$$

Changes outside this range will produce a different dual price (worth per unit).

Using similar computations, you can verify that the dual price for machine 2 capacity is \$2.00/hr and it remains valid for changes (increases or decreases) that move its constraint parallel to itself to any point on the line segment DE, which yields the following limits:

$$\text{Minimum machine 2 capacity [at } D = (4, 0)] = 1 \times 4 + 3 \times 0 = 4 \text{ hr}$$

$$\text{Maximum machine 2 capacity [at } E = (8, 0)] = 1 \times 0 + 3 \times 8 = 24 \text{ hr}$$

The conclusion is that the dual price of \$2.00/hr for machine 2 will remain applicable for the range

$$4 \text{ hr} \leq \text{Machine 2 capacity} \leq 24 \text{ hr}$$

The computed limits for machine 1 and 2 are referred to as **the feasibility ranges**. All software packages provide information about the dual prices and their feasibility ranges. Section 3.6.4 shows how AMPL, Solver, and TORA generate this information.

The dual prices allow making economic decisions about the LP problem, as the following questions demonstrate:

Question 1. If JOBCO can increase the capacity of both machines, which machine should receive higher priority?

The dual prices for machines 1 and 2 are \$14.00/hr and \$2.00/hr. This means that each additional hour of machine 1 will increase revenue by \$14.00, as opposed to only \$2.00 for machine 2. Thus, priority should be given to machine 1.

Question 2. A suggestion is made to increase the capacities of machines 1 and 2 at the additional cost of \$10/hr. Is this advisable?

For machine 1, the additional net revenue per hour is $14.00 - 10.00 = \$4.00$ and for machine 2, the net is $\$2.00 - \$10.00 = -\$8.00$. Hence, only the capacity of machine 1 should be increased.

Question 3. If the capacity of machine 1 is increased from the present 8 hours to 13 hours, how will this increase impact the optimum revenue?

The dual price for machine 1 is \$14.00 and is applicable in the range (2.67, 16) hr. The proposed increase to 13 hours falls within the feasibility range. Hence, the increase in revenue is $\$14.00(13 - 8) = \$70.00$, which means that the total revenue will be increased to (current revenue + change in revenue) $= 128 + 70 = \$198.00$.

Question 4. Suppose that the capacity of machine 1 is increased to 20 hours, how will this increase impact the optimum revenue?

The proposed change is outside the range (2.67, 16) hr for which the dual price of $14.00 remains applicable. Thus, we can only make an immediate conclusion regarding an increase up to 16 hours. Beyond that, further calculations are needed to find the answer (see Chapter 4). Remember that falling outside the feasibility range does *not* mean that the problem has no solution. It only means that we do not have sufficient information to make an *immediate* decision.

Question 5. We know that the change in the optimum objective value equals (dual price $\times$ change in resource) so long as the change in the resource is within the feasibility range. What about the associated optimum values of the variables?

The optimum values of the variables will definitely change. However, the level of information we have from the graphical solution is not sufficient to determine the new values. Section 3.6.2, which treats the sensitivity problem algebraically, provides this detail.

PROBLEM SET 3.6A

1. A company produces two products, *A* and *B*. The unit revenues are $2 and $3, respectively. Two raw materials, *M*1 and *M*2, used in the manufacture of the two products have respective daily availabilities of 8 and 18 units. One unit of *A* uses 2 units of *M*1 and 2 units of *M*2, and 1 unit of *B* uses 3 units of *M*1 and 6 units of *M*2.
   (a) Determine the dual prices of *M*1 and *M*2 and their feasibility ranges.
   (b) Suppose that 4 additional units of *M*1 can be acquired at the cost of 30 cents per unit. Would you recommend the additional purchase?
   (c) What is the most the company should pay per unit of *M*2?
   (d) If *M*2 availability is increased by 5 units, determine the associated optimum revenue.

*2. Wild West produces two types of cowboy hats. A Type 1 hat requires twice as much labor time as a Type 2. If all the available labor time is dedicated to Type 2 alone, the company can produce a total of 400 Type 2 hats a day. The respective market limits for the two types are 150 and 200 hats per day. The revenue is $8 per Type 1 hat and $5 per Type 2 hat.
   (a) Use the graphical solution to determine the number of hats of each type that maximizes revenue.
   (b) Determine the dual price of the production capacity (in terms of the Type 2 hat) and the range for which it is applicable.
   (c) If the daily demand limit on the Type 1 hat is decreased to 120, use the dual price to determine the corresponding effect on the optimal revenue.
   (d) What is the dual price of the market share of the Type 2 hat? By how much can the market share be increased while yielding the computed worth per unit?

Example 3.6-2 (Changes in the Objective Coefficients)

Figure 3.13 shows the graphical solution space of the JOBCO problem presented in Example 3.6-1. The optimum occurs at point C ($x_1 = 3.2, x_2 = 1.6, z = 128$). Changes in revenue units (i.e., objective-function coefficients) will change the slope of z. However, as can be seen from the figure, the optimum solution will remain at point C so long as the objective function lies between lines BF and DE, the two constraints that define the optimum point. This means that there is a range for the coefficients of the objective function that will keep the optimum solution unchanged at C.

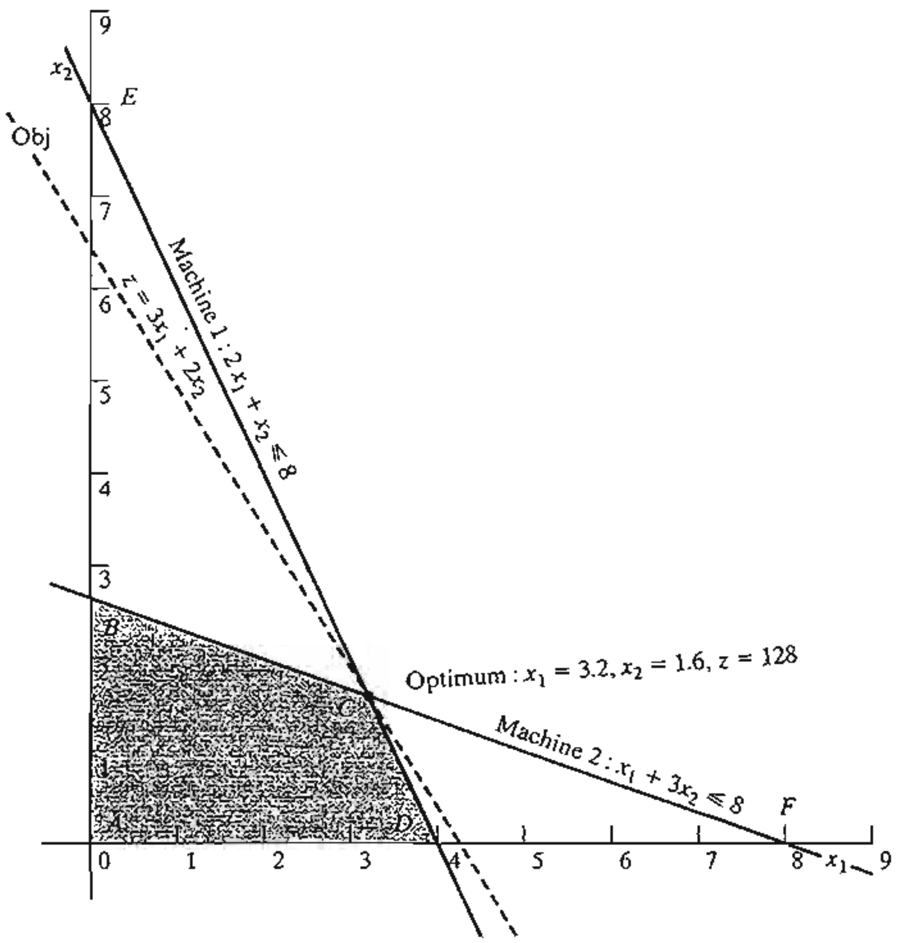


FIGURE 3.13

Graphical sensitivity of optimal solution to changes in the revenue units (coefficients of the objective function)

We can write the objective function in the general format

$$\text{Maximize } z = c_1x_1 + c_2x_2$$

Imagine now that the line z is pivoted at C and that it can rotate clockwise and counterclockwise. The optimum solution will remain at point C so long as $z = c_1x_1 + c_2x_2$ lies between the two lines $x_1 + 3x_2 = 8$ and $2x_1 + x_2 = 8$. This means that the ratio $\frac{c_1}{c_2}$ can vary between $\frac{1}{3}$ and $\frac{2}{1}$, which yields the following condition:

$$\frac{1}{3} \leq \frac{c_1}{c_2} \leq \frac{2}{1} \quad \text{or} \quad .333 \leq \frac{c_1}{c_2} \leq 2$$

This information can provide immediate answers regarding the optimum solution as the following questions demonstrate:

Question 1. Suppose that the unit revenues for products 1 and 2 are changed to \$35 and \$25, respectively. Will the current optimum remain the same?

The new objective function is

$$\text{Maximize } z = 35x_1 + 25x_2$$

The solution at C will remain optimal because $\frac{c_1}{c_2} = \frac{35}{25} = 1.4$ remains within the optimality range $(.333, 2)$. When the ratio falls outside this range, additional calculations are needed to find the new optimum (see Chapter 4). Notice that although the values of the variables at the optimum point C remain unchanged, the optimum value of z changes to $35 \times (3.2) + 25 \times (1.6) = \$152.00$.

Question 2. Suppose that the unit revenue of product 2 is fixed at its current value of $c_2 = \$20.00$. What is the associated range for c_1, the unit revenue for product 1 that will keep the optimum unchanged?

Substituting $c_2 = 20$ in the condition $\frac{1}{3} \leq \frac{c_1}{c_2} \leq 2$, we get

$$\tfrac{1}{3} \times 20 \leq c_1 \leq 2 \times 20$$

Or

$$6.67 \leq c_1 \leq 40$$

This range is referred to as the **optimality range** for c_1, and it implicitly assumes that c_2 is fixed at \$20.00.

We can similarly determine the *optimality range* for c_2 by fixing the value of c_1 at \$30.00. Thus,

$$c_2 \leq 30 \times 3 \text{ and } c_2 \geq \tfrac{30}{2}$$

Or

$$15 \leq c_2 \leq 90$$

As in the case of the right-hand side, all software packages provide the optimality ranges. Section 3.6.4 shows how AMPL, Solver, and TORA generate these results.

Remark. Although the material in this section has dealt only with two variables, the results lay the foundation for the development of sensitivity analysis for the general LP problem in Sections 3.6.2 and 3.6.3.

PROBLEM SET 3.6B

1. Consider Problem 1, Set 3.6a.
   - **(a)** Determine the optimality condition for $\frac{c_A}{c_B}$ that will keep the optimum unchanged.
   - **(b)** Determine the optimality ranges for c_A and c_B, assuming that the other coefficient is kept constant at its present value.
   - **(c)** If the unit revenues c_A and c_B are changed simultaneously to \$5 and \$4, respectively, determine the new optimum solution.
   - **(d)** If the changes in (c) are made one at a time, what can be said about the optimum solution?
2. In the Reddy Mikks model of Example 2.2-1;
   - **(a)** Determine the range for the ratio of the unit revenue of exterior paint to the unit revenue of interior paint.

(b) If the revenue per ton of exterior paint remains constant at \$5000 per ton, determine the maximum unit revenue of interior paint that will keep the present optimum solution unchanged.

(c) If for marketing reasons the unit revenue of interior paint must be reduced to \$3000, will the current optimum production mix change?

***3.** In Problem 2, Set 3.6a:

(a) Determine the optimality range for the unit revenue ratio of the two types of hats that will keep the current optimum unchanged.

(b) Using the information in (b), will the optimal solution change if the revenue per unit is the same for both types?

3.6.2 Algebraic Sensitivity Analysis—Changes in the Right-Hand Side

In Section 3.6.1, we used the graphical solution to determine the *dual prices* (the unit worths of resources) and their feasibility ranges. This section extends the analysis to the general LP model. A numeric example (the TOYCO model) will be used to facilitate the presentation.

Example 3.6-2 (TOYCO Model)

TOYCO assembles three types of toys—trains, trucks, and cars—using three operations. The daily limits on the available times for the three operations are 430, 460, and 420 minutes, respectively, and the revenues per unit of toy train, truck, and car are \$3, \$2, and \$5, respectively. The assembly times per train at the three operations are 1, 3, and 1 minutes, respectively. The corresponding times per train and per car are (2, 0, 4) and (1, 2, 0) minutes (a zero time indicates that the operation is not used).

Letting x_1, x_2, and x_3 represent the daily number of units assembled of trains, trucks, and cars, respectively, the associated LP model is given as:

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\le 430 \text{ (Operation 1)} \\ 3x_1 + 2x_3 &\le 460 \text{ (Operation 2)} \\ x_1 + 4x_2 &\le 420 \text{ (Operation 3)} \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Using x_4, x_5, and x_6 as the slack variables for the constraints of operations 1, 2, and 3, respectively, the optimum tableau is

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

The solution recommends manufacturing 100 trucks and 230 cars but no trains. The associated revenue is $1350.

Determination of Dual Prices. The constraints of the model after adding the slack variables x_4, x_5, and x_6 can be written as follows:

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 430 \quad \text{(Operation 1)} \\ 3x_1 + 2x_3 + x_5 &= 460 \quad \text{(Operation 2)} \\ x_1 + 4x_2 + x_6 &= 420 \quad \text{(Operation 3)} \end{aligned}$$

or

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 430 - x_4 \quad \text{(Operation 1)} \\ 3x_1 + 2x_3 &= 460 - x_5 \quad \text{(Operation 2)} \\ x_1 + 4x_2 &= 420 - x_6 \quad \text{(Operation 3)} \end{aligned}$$

With this representation, the slack variables have the same units (minutes) as the operation times. Thus, we can say that a one-minute *decrease* in the slack variable is equivalent to a one-minute *increase* in the operation time.

We can use the information above to determine the *dual prices* from the z-equation in the optimal tableau:

$$z + 4x_1 + x_4 + 2x_5 + 0x_6 = 1350$$

This equation can be written as

$$\begin{aligned} z &= 1350 - 4x_1 - x_4 - 2x_5 - 0x_6 \\ &= 1350 - 4x_1 + 1(-x_4) + 2(-x_5) + 0(-x_6) \end{aligned}$$

Given that a *decrease* in the value of a slack variable is equivalent to an *increase* in its operation time, we get

$$\begin{aligned} z = 1350 - 4x_1 &+ 1 \times \text{(increase in operation 1 time)} \\ &+ 2 \times \text{(increase in operation 2 time)} \\ &+ 0 \times \text{(increase in operation 3 time)} \end{aligned}$$

This equation reveals that (1) a one-minute increase in operation 1 time increases z by \$1, (2) a one-minute increase in operation 2 time increases z by \$2, and (3) a one-minute increase in operation 3 time does not change z.

To summarize, the z-row in the optimal tableau:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350

yields directly the dual prices, as the following table shows:

Resource	Slack variable	Optimal z-equation coefficient of slack variable	Dual price
Operation 1	x_4	1	\$1/min
Operation 2	x_5	2	\$2/min
Operation 3	x_6	0	\$0/min

The zero dual price for operation 3 means that there is no economic advantage in allocating more production time to this operation. The result makes sense because the resource is already abundant, as is evident by the fact that the slack variable associated with Operation 3 is positive (= 20) in the optimum solution. As for each of Operations 1 and 2, a one minute increase will improve revenue by \$1 and \$2, respectively. The dual prices also indicate that, when allocating additional resources, Operation 2 may be given higher priority because its dual price is twice as much as that of Operation 1.

The computations above show how the dual prices are determined from the optimal tableau for $\leq$ constraints. For $\geq$ constraints, the same idea remains applicable except that the dual price will assume the opposite sign of that associated with the $\leq$ constraint. As for the case where the constraint is an equation, the determination of the dual price from the optimal simplex tableau requires somewhat "involved" calculations as will be shown in Chapter 4.

Determination of the Feasibility Ranges. Having determined the dual prices, we show next how the *feasibility ranges* in which they remain valid are determined. Let D_1, D_2, and D_3 be the changes (positive or negative) in the daily manufacturing time allocated to operations 1, 2, and 3, respectively. The model can be written as follows:

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned}
x_1 + 2x_2 + x_3 &\leq 430 + D_1 \quad \text{(Operation 1)} \\
3x_1 \qquad + 2x_3 &\leq 460 + D_2 \quad \text{(Operation 2)} \\
x_1 + 4x_2 \qquad &\leq 420 + D_3 \quad \text{(Operation 3)} \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

We will consider the general case of making the changes simultaneously. The special cases of making change one at a time are derived from these results.

The procedure is based on recomputing the optimum simplex tableau with the modified right-hand side and then deriving the conditions that will keep the solution feasible—that is, the right-hand side of the optimum tableau remains nonnegative. To show how the right-hand side is recomputed, we start by modifying the *Solution* column of the starting tableau using the new right-hand sides: $430 + D_1$, $460 + D_2$, and $420 + D_3$. The starting tableau will thus appear as

							Solution			
Basic	x_1	x_2	x_3	x_4	x_5	x_6	*RHS*	D_1	D_2	D_3
z	-3	-2	-5	0	0	0	0	0	0	0
x_4	1	2	1	1	0	0	430	1	0	0
x_5	3	0	2	0	1	0	460	0	1	0
x_6	1	4	0	0	0	1	420	0	0	1

The columns under D_1, D_2, and D_3 are identical to those under the starting basic columns x_4, x_5, and x_6. This means that when we carry out the *same* simplex iterations as in the *original* model, the columns in the two groups must come out identical as well. Effectively, the new optimal tableau will become

							Solution			
Basic	x_1	x_2	x_3	x_4	x_5	x_6	*RHS*	D_1	D_2	D_3
z	4	0	0	1	2	0	1350	1	2	0
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100	$\frac{1}{2}$	$-\frac{1}{4}$	0
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	0	$\frac{1}{2}$	0
x_6	2	0	0	-2	1	1	20	-2	1	1

The new optimum tableau provides the following optimal solution:

$$z = 1350 + D_1 + 2D_2$$
$$x_2 = 100 + \tfrac{1}{2}D_1 - \tfrac{1}{4}D_2$$
$$x_3 = 230 + \tfrac{1}{2}D_2$$
$$x_6 = 20 - 2D_1 + D_2 + D_3$$

Interestingly, as shown earlier, the new z-value confirms that the dual prices for operations 1, 2, and 3 are 1, 2, and 0, respectively.

The current solution remains feasible so long as all the variables are nonnegative, which leads to the following **feasibility conditions**:

$$x_2 = 100 + \tfrac{1}{2}D_1 - \tfrac{1}{4}D_2 \geq 0$$
$$x_3 = 230 + \tfrac{1}{2}D_2 \geq 0$$
$$x_6 = 20 - 2D_1 + D_2 + D_3 \geq 0$$

Any simultaneous changes D_1, D_2, and D_3 that satisfy these inequalities will keep the solution feasible. If all the conditions are satisfied, then the new optimum solution can be found through direct substitution of D_1, D_2, and D_3 in the equations given above.

To illustrate the use of these conditions, suppose that the manufacturing time available for operations 1, 2, and 3 are 480, 440, and 410 minutes respectively. Then, $D_1 = 480 - 430 = 50$, $D_2 = 440 - 460 = -20$, and $D_3 = 410 - 420 = -10$. Substituting in the feasibility conditions, we get

$$x_2 = 100 + \tfrac{1}{2}(50) - \tfrac{1}{4}(-20) = 130 > 0 \quad \text{(feasible)}$$
$$x_3 = 230 + \tfrac{1}{2}(-20) = 220 > 0 \quad \text{(feasible)}$$
$$x_6 = 20 - 2(50) + (-20) + (-10) = -110 < 0 \quad \text{(infeasible)}$$

The calculations show that $x_6 < 0$, hence the current solution does not remain feasible. Additional calculations will be needed to find the new solution. These calculations are discussed in Chapter 4 as part of the post-optimal analysis.

Alternatively, if the changes in the resources are such that $D_1 = -30$, $D_2 = -12$, and $D_3 = 10$, then

$$x_2 = 100 + \tfrac{1}{2}(-30) - \tfrac{1}{4}(-12) = 88 > 0 \quad \text{(feasible)}$$
$$x_3 = 230 + \tfrac{1}{2}(-12) = 224 > 0 \quad \text{(feasible)}$$
$$x_6 = 20 - 2(-30) + (-12) + (10) = 78 > 0 \quad \text{(feasible)}$$

The new feasible solution is $x_1 = 88$, $x_3 = 224$, and $x_6 = 68$ with $z = 3(0) + 2(88) + 5(224) = \$1296$. Notice that the optimum objective value can also be computed as $z = 1350 + 1(-30) + 2(-12) = \$1296$.

The given conditions can be specialized to produce the individual *feasibility ranges* that result from changing the resources *one at a time* (as defined in Section 3.6.1).

Case 1. Change in operation 1 time from 460 *to* 460 + D_1 *minutes.* This change is equivalent to setting $D_2 = D_3 = 0$ in the simultaneous conditions, which yields

$$\left.\begin{aligned} x_2 &= 100 + \tfrac{1}{2}D_1 \geq 0 \Rightarrow D_1 \geq -200 \\ x_3 &= 230 > 0 \\ x_6 &= 20 - 2D_1 \geq 0 \Rightarrow D_1 \leq 10 \end{aligned}\right\} \Rightarrow -200 \leq D_1 \leq 10$$

Case 2. Change in operation 2 time from 430 *to* 430 + D_2 *minutes.* This change is equivalent to setting $D_1 = D_3 = 0$ in the simultaneous conditions, which yields

$$\left.\begin{aligned} x_2 &= 100 - \tfrac{1}{4}D_2 \geq 0 \Rightarrow D_2 \leq 400 \\ x_3 &= 230 + \tfrac{1}{2}D_2 \geq 0 \Rightarrow D_2 \geq -460 \\ x_6 &= 20 + D_2 \geq 0 \quad \Rightarrow D_2 \geq -20 \end{aligned}\right\} \Rightarrow -20 \leq D_2 \leq 400$$

Case 3. Change in operation 3 time from 420 *to* 420 + D_3 *minutes.* This change is equivalent to setting $D_1 = D_2 = 0$ in the simultaneous conditions, which yields

$$\left.\begin{aligned} x_2 &= 100 > 0 \\ x_3 &= 230 > 0 \\ x_6 &= 20 + D_3 \geq 0 \end{aligned}\right\} \Rightarrow -20 \leq D_3 < \infty$$

We can now summarize the dual prices and their feasibility ranges for the TOYCO model as follows:[3]

			Resource amount (minutes)		
Resource	Dual price	Feasibility range	*Minimum*	*Current*	*Maximum*
Operation 1	1	$-200 \leq D_1 \leq 10$	230	430	440
Operation 2	2	$-20 \leq D_2 \leq 400$	440	440	860
Operation 3	0	$-20 \leq D_3 < \infty$	400	420	∞

It is important to notice that the dual prices will remain applicable for any *simultaneous* changes that keep the solution feasible, even if the changes violate the individual ranges. For example, the changes $D_1 = 30$, $D_2 = -12$, and $D_3 = 100$, will keep the solution feasible even though $D_1 = 30$ violates the feasibility range $-200 \leq D_1 \leq 10$, as the following computations show:

$$x_2 = 100 + \tfrac{1}{2}(30) - \tfrac{1}{4}(-12) = 118 > 0 \quad \text{(feasible)}$$

$$x_3 = 230 + \tfrac{1}{2}(-12) = 224 > 0 \quad \text{(feasible)}$$

$$x_6 = 20 - 2(30) + (-12) + (100) = 48 > 0 \quad \text{(feasible)}$$

This means that the dual prices will remain applicable, and we can compute the new optimum objective value from the dual prices as $z = 1350 + 1(30) + 2(-12) + 0(100) = \$1356$

The results above can be summarized as follows:

1. The dual prices remain valid so long as the changes D_i, $i = 1, 2, \ldots, m$, in the right-hand sides of the constraints satisfy all the feasibility conditions when the changes are simultaneous or fall within the feasibility ranges when the changes are made individually.

2. For other situations where the dual prices are not valid because the simultaneous feasibility conditions are not satisfied or because the individual feasibility ranges are violated, the recourse is to either re-solve the problem with the new values of D_i or apply the post-optimal analysis presented in Chapter 4.

PROBLEM SET 3.6C[4]

1. In the TOYCO model, suppose that the changes D_1, D_2, and D_3 are made *simultaneously* in the three operations.
   (a) If the availabilities of operations 1, 2, and 3 are changed to 438, 500, and 410 minutes, respectively, use the simultaneous conditions to show that the current basic solution

[3] Available LP packages usually present this information as standard output. Practically none provide the case of simultaneous conditions, presumably because its display is cumbersome, particularly for large LPs.

[4] In this problem set, you may find it convenient to generate the optimal simplex tableau with TORA.

remains feasible, and determine the change in the optimal revenue by using the optimal dual prices.

(b) If the availabilities of the three operations are changed to 460, 440, and 380 minutes, respectively, use the simultaneous conditions to show that the current basic solution becomes infeasible.

*2. Consider the TOYCO model.

(a) Suppose that any additional time for operation 1 beyond its current capacity of 430 minutes per day must be done on an overtime basis at $50 an hour. The hourly cost includes both labor and the operation of the machine. Is it economically advantageous to use overtime with operation 1?

(b) Suppose that the operator of operation 2 has agreed to work 2 hours of overtime daily at $45 an hour. Additionally, the cost of the operation itself is $10 an hour. What is the net effect of this activity on the daily revenue?

(c) Is overtime needed for operation 3?

(d) Suppose that the daily availability of operation 1 is increased to 440 minutes. Any overtime used beyond the current maximum capacity will cost $40 an hour. Determine the new optimum solution, including the associated net revenue.

(e) Suppose that the availability of operation 2 is decreased by 15 minutes a day and that the hourly cost of the operation during regular time is $30. Is it advantageous to decrease the availability of operation 2?

3. A company produces three products, A, B, and C. The sales volume for A is at least 50% of the total sales of all three products. However, the company cannot sell more than 75 units of A per day. The three products use one raw material, of which the maximum daily availability is 240 lb. The usage rates of the raw material are 2 lb per unit of A, 4 lb per unit of B, and 3 lb per unit of C. The unit prices for A, B, and C are $20, $50, and $35, respectively.

(a) Determine the optimal product mix for the company.

(b) Determine the dual price of the raw material resource and its allowable range. If available raw material is increased by 120 lb, determine the optimal solution and the change in total revenue using the dual price.

(c) Use the dual price to determine the effect of changing the maximum demand for product A by ± 10 units.

4. A company that operates 10 hours a day manufactures three products on three sequential processes. The following table summarizes the data of the problem:

	Minutes per unit			
Product	*Process 1*	*Process 2*	*Process 3*	Unit price
1	10	6	8	$4.50
2	5	8	10	$5.00
3	6	9	12	$4.00

(a) Determine the optimal product mix.

(b) Use the dual prices to prioritize the three processes for possible expansion.

(c) If additional production hours can be allocated, what would be a fair cost per additional hour for each process?

5. The Continuing Education Division at the Ozark Community College offers a total of 30 courses each semester. The courses offered are usually of two types: practical, such as woodworking, word processing, and car maintenance; and humanistic, such as history, music, and fine arts. To satisfy the demands of the community, at least 10 courses of each type must be offered each semester. The division estimates that the revenues of offering practical and humanistic courses are approximately \$1500 and \$1000 per course, respectively.
   (a) Devise an optimal course offering for the college.
   (b) Show that the dual price of an additional course is \$1500, which is the same as the revenue per practical course. What does this result mean in terms of offering additional courses?
   (c) How many more courses can be offered while guaranteeing that each will contribute \$1500 to the total revenue?
   (d) Determine the change in revenue resulting from increasing the minimum requirement of humanistics by one course.

*6. Show & Sell can advertise its products on local radio and television (TV), or in newspapers. The advertising budget is limited to \$10,000 a month. Each minute of advertising on radio costs \$15 and each minute on TV costs \$300. A newspaper ad costs \$50. Show & Sell likes to advertise on radio at least twice as much as on TV. In the meantime, the use of at least 5 newspaper ads and no more than 400 minutes of radio advertising a month is recommended. Past experience shows that advertising on TV is 50 times more effective than on radio and 10 times more effective than in newspapers.
   (a) Determine the optimum allocation of the budget to the three media.
   (b) Are the limits set on radio and newspaper advertising justifiable economically?
   (c) If the monthly budget is increased by 50%, would this result in a proportionate increase in the overall effectiveness of advertising?

7. The Burroughs Garment Company manufactures men's shirts and women's blouses for Walmark Discount Stores. Walmark will accept all the production supplied by Burroughs. The production process includes cutting, sewing, and packaging. Burroughs employs 25 workers in the cutting department, 35 in the sewing department, and 5 in the packaging department. The factory works one 8-hour shift, 5 days a week. The following table gives the time requirements and prices per unit for the two garments:

	Minutes per unit			
Garment	*Cutting*	*Sewing*	*Packaging*	Unit price (\$)
Shirts	20	70	12	8.00
Blouses	60	60	4	12.00

   (a) Determine the optimal weekly production schedule for Burroughs.
   (b) Determine the worth of one hour of cutting, sewing, and packaging in terms of the total revenue.
   (c) If overtime can be used in cutting and sewing, what is the maximum hourly rate Burroughs should pay for overtime?

8. ChemLabs uses raw materials I and II to produce two domestic cleaning solutions, A and B. The daily availabilities of raw materials I and II are 150 and 145 units, respectively. One unit of solution A consumes .5 unit of raw material I and .6 unit of raw material II, and one unit of solution B uses .5 unit of raw material I and .4 unit of raw material II. The

prices per unit of solutions A and B are \$8 and \$10, respectively. The daily demand for solution A lies between 30 and 150 units, and that for solution B between 40 and 200 units.

(a) Find the optimal amounts of A and B that ChemLab should produce.

(b) Use the dual prices to determine which demand limits on products A and B should be relaxed to improve profitability.

(c) If additional units of raw material can be acquired at \$20 per unit, is this advisable? Explain.

(d) A suggestion is made to increase raw material II by 25% to remove a bottleneck in production. Is this advisable? Explain.

9. An assembly line consisting of three consecutive workstations produces two radio models: DiGi-1 and DiGi-2. The following table provides the assembly times for the three workstations.

	Minutes per unit	
Workstation	*DiGi-1*	*DiGi-2*
1	6	4
2	5	4
3	4	6

The daily maintenance for workstations 1, 2, and 3 consumes 10%, 14%, and 12%, respectively, of the maximum 480 minutes available for each workstation each day.

(a) The company wishes to determine the optimal product mix that will minimize the idle (or unused) times in the three workstations. Determine the optimum utilization of the workstations. [*Hint*: Express the sum of the idle times (slacks) for the three operations in terms of the original variables.]

(b) Determine the worth of decreasing the daily maintenance time for each workstation by 1 percentage point.

(c) It is proposed that the operation time for all three workstations be increased to 600 minutes per day at the additional cost of \$1.50 per minute. Can this proposal be improved?

10. The Gutchi Company manufactures purses, shaving bags, and backpacks. The construction of the three products requires leather and synthetics, with leather being the limiting raw material. The production process uses two types of skilled labor: sewing and finishing. The following table gives the availability of the resources, their usage by the three products, and the prices per unit.

	Resource requirements per unit			
Resource	*Purse*	*Bag*	*Backpack*	Daily availability
Leather (ft^2)	2	1	3	42
Sewing (hr)	2	1	2	40
Finishing (hr)	1	.5	1	45
Price (\$)	24	22	45	

Formulate the problem as a linear program and find the optimum solution. Next, indicate whether the following changes in the resources will keep the current solution feasible.

For the cases where feasibility is maintained, determine the new optimum solution (values of the variables and the objective function).

(a) Available leather is increased to 45 ft^2.

(b) Available leather is decreased by 1 ft^2.

(c) Available sewing hours are changed to 38 hours.

(d) Available sewing hours are changed to 46 hours.

(e) Available finishing hours are decreased to 15 hours.

(f) Available finishing hours are increased to 50 hours.

(g) Would you recommend hiring an additional sewing worker at $15 an hour?

11. HiDec produces two models of electronic gadgets that use resistors, capacitors, and chips. The following table summarizes the data of the situation:

	Unit resource requirements		
Resource	*Model 1 (units)*	*Model 2 (units)*	Maximum availability (units)
Resistor	2	3	1200
Capacitor	2	1	1000
Chips	0	4	800
Unit price ($)	3	4	

Let x_1 and x_2 be the amounts produced of Models 1 and 2, respectively. Following are the LP model and its associated optimal simplex tableau.

$$\text{Maximize } z = 3x_1 + 4x_2$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 &\le 1200 \quad \text{(Resistors)} \\ 2x_1 + x_2 &\le 1000 \quad \text{(Capacitors)} \\ 4x_2 &\le 800 \quad \text{(Chips)} \\ x_1, x_2 &\ge 0 \end{aligned}$$

Basic	x_1	x_2	s_1	s_2	s_3	Solution
z	0	0	$\frac{5}{4}$	$\frac{1}{4}$	0	1750
x_1	1	0	$-\frac{1}{4}$	$\frac{3}{4}$	0	450
s_3	0	0	-2	2	1	400
x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	100

***(a)** Determine the status of each resource.

***(b)** In terms of the optimal revenue, determine the dual prices for the resistors, capacitors, and chips.

(c) Determine the feasibility ranges for the dual prices obtained in (b).

(d) If the available number of resistors is increased to 1300 units, find the new optimum solution.

*(e) If the available number of chips is reduced to 350 units, will you be able to determine the new optimum solution directly from the given information? Explain.

(f) If the availability of capacitors is limited by the feasibility range computed in (c), determine the corresponding range of the optimal revenue and the corresponding ranges for the numbers of units to be produced of Models 1 and 2.

(g) A new contractor is offering to sell HiDec additional resistors at 40 cents each, but only if HiDec would purchase at least 500 units. Should HiDec accept the offer?

12. *The 100% feasibility rule.* A simplified rule based on the *individual* changes $D_1, D_2, \ldots,$ and D_m in the right-hand side of the constraints can be used to test whether or not *simultaneous* changes will maintain the feasibility of the current solution. Assume that the right-hand side b_i of constraint i is changed to $b_i + D_i$ *one at a time*, and that $p_i \le D_i \le q_i$ is the corresponding feasibility range obtained by using the procedure in Section 3.6.2. By definition, we have $p_i \le 0$ ($q_i \ge 0$) because it represents the maximum allowable decrease (increase) in b_i. Next, define r_i to equal $\frac{D_i}{p_i}$ if D_i is negative and $\frac{D_i}{q_i}$ if D_i is positive. By definition, we have $0 \le r_i \le 1$. The 100% rule thus says that, given the changes $D_1, D_2, \ldots,$ and D_m, then a *sufficient* (but not necessary) condition for the current solution to remain feasible is that $r_1 + r_2 + \cdots + r_m \le 1$. If the condition is not satisfied, then the current solution may or may not remain feasible. The rule is not applicable if D_i falls outside the range (p_i, q_i).

In reality, the 100% rule is too weak to be consistently useful. Even in the cases where feasibility can be confirmed, we still need to obtain the new solution using the regular simplex feasibility conditions. Besides, the direct calculations associated with simultaneous changes given in Section 3.6.2 are straightforward and manageable.

To demonstrate the weakness of the rule, apply it to parts (a) and (b) of Problem 1 in this set. The rule fails to confirm the feasibility of the solution in (a) and does not apply in (b) because the changes in D_i are outside the admissible ranges. Problem 13 further demonstrates this point.

13. Consider the problem

$$\text{Maximize } z = x_1 + x_2$$

subject to

$$2x_1 + x_2 \le 6$$
$$x_1 + 2x_2 \le 6$$
$$x_1 + x_2 \ge 0$$

(a) Show that the optimal basic solution includes both x_1 and x_2 and that the feasibility ranges for the two constraints, considered one at a time, are $-3 \le D_1 \le 6$ and $-3 \le D_2 \le 6$.

*(b) Suppose that the two resources are increased simultaneously by $\Delta > 0$ each. First, show that the basic solution remains feasible for all $\Delta > 0$. Next, show that the 100% rule will confirm feasibility only if the increase is in the range $0 < \Delta \le 3$ units. Otherwise, the rule fails for $3 < \Delta \le 6$ and does not apply for $\Delta > 6$.

3.6.3 Algebraic Sensitivity Analysis—Objective Function

In Section 3.6.1, we used graphical sensitivity analysis to determine the conditions that will maintain the optimality of a two-variable LP solution. In this section, we extend these ideas to the general LP problem.

Definition of Reduced Cost. To facilitate the explanation of the objective function sensitivity analysis, first we need to define *reduced costs*. In the TOYCO model (Example 3.6-2), the objective z-equation in the optimal tableau is

$$z + 4x_1 + x_4 + 2x_5 = 1350$$

or

$$z = 1350 - 4x_1 - x_4 - 2x_5$$

The optimal solution does not recommend the production of toy trains ($x_1 = 0$). This recommendation is confirmed by the information in the z-equation because each unit increase in x_1 above its current zero level will decrease the value of z by \$4 — namely, $z = 1350 - 4 \times (1) - 1 \times (0) - 2 \times (0) = \$1346$.

We can think of the coefficient of x_1 in the z-equation (= 4) as a unit *cost* because it causes a reduction in the revenue z. But where does this "cost" come from? We know that x_1 has a unit revenue of \$3 in the original model. We also know that each toy train consumes resources (operations time), which in turn incur cost. Thus, the "attractiveness" of x_1 from the standpoint of optimization depends on the relative values of the revenue per unit and the cost of the resources consumed by one unit. This relationship is formalized in the LP literature by defining the reduced cost as

$$\begin{pmatrix}\text{Reduced cost}\\ \text{per unit}\end{pmatrix} = \begin{pmatrix}\text{Cost of consumed}\\ \text{resources per unit}\end{pmatrix} - (\text{Revenue per unit})$$

To appreciate the significance of this definition, in the original TOYCO model the revenue per unit for toy trucks (= \$2) is less than that for toy trains (= \$3). Yet the optimal solution elects to manufacture toy trucks ($x_2 = 100$ units) and no toy trains ($x_1 = 0$). The reason for this (seemingly nonintuitive) result is that the unit cost of the resources used by toy trucks (i.e., operations time) is smaller than its unit price. The opposite applies in the case of toy trains.

With the given definition of *reduced cost* we can now see that an unprofitable variable (such as x_1) can be made profitable in two ways:

1. By increasing the unit revenue.
2. By decreasing the unit cost of consumed resources.

In most real-life situations, the price per unit may not be a viable option because its value is dictated by market conditions. The real option then is to reduce the consumption of resources, perhaps by making the production process more efficient, as will be shown in Chapter 4.

Determination of the Optimality Ranges. We now turn our attention to determining the conditions that will keep an optimal solution unchanged. The presentation is based on the definition of *reduced cost*.

In the TOYCO model, let d_1, d_2, and d_3 represent the change in unit revenues for toy trucks, trains, and cars, respectively. The objective function then becomes

$$\text{Maximize } z = (3 + d_1)x_1 + (2 + d_2)x_2 + (5 + d_3)x_3$$

As we did for the right-hand side sensitivity analysis in Section 3.6.2, we will first deal with the general situation in which all the coefficients of the objective function are changed *simultaneously* and then specialize the results to the one-at-a-time case.

With the simultaneous changes, the z-row in the starting tableau appears as:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$-3 - d_1$	$-2 - d_2$	$-5 - d_3$	0	0	0	0

When we generate the simplex tableaus using the same sequence of entering and leaving variables in the original model (before the changes d_j are introduced), the optimal iteration will appear as follows (convince yourself that this is indeed the case by carrying out the simplex row operations):

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1$	0	0	$1 + \frac{1}{2}d_2$	$2 - \frac{1}{4}d_2 + \frac{1}{2}d_3$	0	$1350 + 100d_2 + 230d_3$
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	$-\frac{1}{4}$	0	0	-2	1	1	20

The new optimal tableau is exactly the same as in the *original* optimal tableau except that the *reduced costs* (z-equation coefficients) have changed. This means that *changes in the objective-function coefficients can affect the optimality of the problem only.*

You really do not need to carry out the row operation to compute the new reduced costs. An examination of the new z-row shows that the coefficients of d_j are taken directly from the constraint coefficients of the optimum tableau. A convenient way for computing the new reduced cost is to add a new top row and a new leftmost column to the optimum tableau, as shown by the shaded areas below. The entries in the top row are the change d_j associated with each variable. For the leftmost column, the entries are 1 in the z-row and the associated d_j in the row of each basic variable. Keep in mind that $d_j = 0$ for the slack variables.

		d_1	d_2	d_3	0	0	0	
	Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
1	z	4	0	0	1	2	0	1350
d_2	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
d_3	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
0	x_6	2	0	0	-2	1	1	20

Now, to compute the new reduced cost for any variable (or the value of z), multiply the elements of its column by the corresponding elements in the leftmost column, add them up, and subtract the top-row element from the sum. For example, for x_1, we have

Left column	x_1	(x_1-column × left-column)
	d_1	
1	4	4×1
d_2	$-\frac{1}{4}$	$-\frac{1}{4}d_2$
d_3	$\frac{3}{2}$	$\frac{3}{2}d_3$
0	2	2×0
Reduced cost for $x_1 = 4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1$		

Note that the application of these computations to the *basic* variables will always produce a zero reduced cost, a proven theoretical result. Also, applying the same rule to the *Solution* column produces $z = 1350 + 100d_2 + 230d_3$.

Because we are dealing with a maximization problem, the current solution remains optimal so long as the new reduced costs (z-equation coefficients) remain nonnegative for all the nonbasic variables. We thus have the following **optimality conditions** corresponding to nonbasic x_1, x_4, and x_5:

$$4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1 \geq 0$$

$$1 + \frac{1}{2}d_2 \geq 0$$

$$2 - \frac{1}{4}d_2 + \frac{1}{2}d_3 \geq 0$$

These conditions must be satisfied *simultaneously* to maintain the optimality of the current optimum.

To illustrate the use of these conditions, suppose that the objective function of TOYCO is changed from

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

to

$$\text{Maximize } z = 2x_1 + x_2 + 6x_3$$

Then, $d_1 = 2 - 3 = -\$1$, $d_2 = 1 - 2 = -\$1$, and $d_3 = 6 - 5 = \$1$. Substitution in the given conditions yields

$$4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1 = 4 - \frac{1}{4}(-1) + \frac{3}{2}(1) - (-1) = 6.75 > 0 \text{ (satisfied)}$$

$$1 + \frac{1}{2}d_2 = 1 + \frac{1}{2}(-1) = .5 > 0 \qquad \text{(satisfied)}$$

$$2 - \frac{1}{4}d_2 + \frac{1}{2}d_3 = 2 - \frac{1}{4}(-1) + \frac{1}{2}(1) = 2.75 > 0 \qquad \text{(satisfied)}$$

The results show that the proposed changes will keep the current solution ($x_1 = 0$, $x_2 = 100$, $x_3 = 230$) optimal. Hence no further calculations are needed, except that the objective value will change to $z = 1350 + 100d_2 + 230d_3 = 1350 + 100 \times -1 + 230 \times 1 = \$1480$. If any of the conditions is not satisfied, a new solution must be determined (see Chapter 4).

The discussion so far has dealt with the maximization case. The only difference in the minimization case is that the reduced costs (z-equations coefficients) must be ≤ 0 to maintain optimality.

The general optimality conditions can be used to determine the special case where the changes d_j occur *one at a time* instead of simultaneously. This analysis is equivalent to considering the following three cases:

1. Maximize $z = (3 + d_1)x_1 + 2x_2 + 5x_3$
2. Maximize $z = 3x_1 + (2 + d_2)x_2 + 5x_3$
3. Maximize $z = 3x_1 + 2x_2 + (5 + d_3)x_3$

The individual conditions can be accounted for as special cases of the simultaneous case.[5]

Case 1. Set $d_2 = d_3 = 0$ in the simultaneous conditions, which gives

$$4 - d_1 \geq 0 \Rightarrow -\infty < d_1 \leq 4$$

Case 2. Set $d_1 = d_3 = 0$ in the simultaneous conditions, which gives

$$\left.\begin{aligned} 4 - \tfrac{1}{4}d_2 \geq 0 &\Rightarrow d_2 \leq 16 \\ 1 + \tfrac{1}{2}d_2 \geq 0 &\Rightarrow d_2 \geq -2 \\ 2 - \tfrac{1}{4}d_2 \geq 0 &\Rightarrow d_2 \leq 8 \end{aligned}\right\} \Rightarrow -2 \leq d_2 \leq 8$$

Case 3. Set $d_1 = d_2 = 0$ in the simultaneous conditions, which gives

$$\left.\begin{aligned} 4 + \tfrac{3}{2}d_3 \geq 0 &\Rightarrow d_3 \geq -\tfrac{8}{3} \\ 2 + \tfrac{1}{2}d_3 \geq 0 &\Rightarrow d_3 \geq -4 \end{aligned}\right\} \Rightarrow -\tfrac{8}{3} \leq d_3 < \infty$$

The given individual conditions can be translated in terms of the total unit revenue. For example, for toy trucks (variable x_2), the total unit revenue is $2 + d_2$ and the associated condition $-2 \leq d_2 \leq 8$ translates to

$$2 + (-2) \leq 2 + d_2 \leq 2 + 8$$

or

$$\$0 \leq (\text{Unit revenue of toy truck}) \leq \$10$$

This condition assumes that the unit revenues for toy trains and toy cars remain fixed at \$3 and \$5, respectively.

The allowable range (\$0, \$10) indicates that the unit revenue of toy trucks (variable x_2) can be as low as \$0 or as high as \$10 without changing the current optimum, $x_1 = 0, x_2 = 100, x_3 = 230$. The total revenue will change to $1350 + 100d_2$, however.

[5]The individual ranges are standard outputs in all LP software. Simultaneous conditions usually are not part of the output, presumably because they are cumbersome for large problems.

It is important to notice that the changes d_1, d_2, and d_3 may be within their allowable individual ranges without satisfying the simultaneous conditions, and vice versa. For example, consider

$$\text{Maximize } z = 6x_1 + 8x_2 + 3x_3$$

Here $d_1 = 6 - 3 = \$3$, $d_2 = 8 - 2 = \$6$, and $d_3 = 3 - 5 = -\$2$, which are all within the permissible individual ranges $(-\infty < d_1 \leq 4, -2 \leq d_2 \leq 8$, and $-\frac{8}{3} \leq d_3 < \infty)$. However, the corresponding simultaneous conditions yield

$$4 - \tfrac{1}{4}d_2 + \tfrac{3}{2}d_3 - d_1 = 4 - \tfrac{1}{4}(6) + \tfrac{3}{2}(-2) - 3 = -3.5 < 0 \text{ (not satisfied)}$$

$$1 + \tfrac{1}{2}d_2 = 1 + \tfrac{1}{2}(6) = 4 > 0 \qquad \text{(satisfied)}$$

$$2 - \tfrac{1}{4}d_2 + \tfrac{1}{2}d_3 = 2 - \tfrac{1}{4}(6) + \tfrac{1}{2}(-2) = -.5 < 0 \qquad \text{(not satisfied)}$$

The results above can be summarized as follows:

1. The optimal values of the variables remain unchanged so long as the changes d_j, $j = 1, 2, \ldots, n$, in the objective function coefficients satisfy all the optimality conditions when the changes are simultaneous or fall within the optimality ranges when a change is made individually.
2. For other situations where the simultaneous optimality conditions are not satisfied or the individual feasibility ranges are violated, the recourse is to either resolve the problem with the new values of d_j or apply the post-optimal analysis presented in Chapter 4.

PROBLEM SET 3.6D[6]

1. In the TOYCO model, determine if the current solution will change in each of the following cases:
   (i) $z = 2x_1 + x_2 + 4x_3$
   (ii) $z = 3x_1 + 6x_2 + x_3$
   (iii) $z = 8x_1 + 3x_2 + 9x_3$

*2. B&K grocery store sells three types of soft drinks: the brand names A1 Cola and A2 Cola and the cheaper store brand BK Cola. The price per can for A1, A2, and BK are 80, 70, and 60 cents, respectively. On the average, the store sells no more than 500 cans of all colas a day. Although A1 is a recognized brand name, customers tend to buy more A2 and BK because they are cheaper. It is estimated that at least 100 cans of A1 are sold daily and that A2 and BK combined outsell A1 by a margin of at least 4:2.
   (a) Show that the optimum solution does not call for selling the A3 brand.
   (b) By how much should the price per can of A3 be increased to be sold by B&K?
   (c) To be competitive with other stores, B&K decided to lower the price on all three types of cola by 5 cents per can. Recompute the reduced costs to determine if this promotion will change the current optimum solution.

[6]In this problem set, you may find it convenient to generate the optimal simplex tableau with TORA.

3. Baba Furniture Company employs four carpenters for 10 days to assemble tables and chairs. It takes 2 person-hours to assemble a table and .5 person-hour to assemble a chair. Customers usually buy one table and four to six chairs. The prices are \$135 per table and \$50 per chair. The company operates one 8-hour shift a day.
   (a) Determine the 10-day optimal production mix.
   (b) If the present unit prices per table and chair are each reduced by 10%, use sensitivity analysis to determine if the optimum solution obtained in (a) will change.
   (c) If the present unit prices per table and chair are changed to \$120 and \$25, will the solution in (a) change?
4. The Bank of Elkins is allocating a maximum of \$200,000 for personal and car loans during the next month. The bank charges 14% for personal loans and 12% for car loans. Both types of loans are repaid at the end of a 1-year period. Experience shows that about 3% of personal loans and 2% of car loans are not repaid. The bank usually allocates at least twice as much to car loans as to personal loans.
   (a) Determine the optimal allocation of funds between the two loans and the net rate of return on all the loans.
   (b) If the percentages of personal and car loans are changed to 4% and 3%, respectively, use sensitivity analysis to determine if the optimum solution in (a) will change.

*5. Electra produces four types of electric motors, each on a separate assembly line. The respective capacities of the lines are 500, 500, 800, and 750 motors per day. Type 1 motor uses 8 units of a certain electronic component, type 2 motor uses 5 units, type 3 motor uses 4 units, and type 4 motor uses 6 units. The supplier of the component can provide 8000 pieces a day. The prices per motor for the respective types are \$60, \$40, \$25, \$30.
   (a) Determine the optimum daily production mix.
   (b) The present production schedule meets Electra's needs. However, because of competition, Electra may need to lower the price of type 2 motor. What is the most reduction that can be effected without changing the present production schedule?
   (c) Electra has decided to slash the price of all motor types by 25%. Use sensitivity analysis to determine if the optimum solution remains unchanged.
   (d) Currently, type 4 motor is not produced. By how much should its price be increased to be included in the production schedule?

6. Popeye Canning is contracted to receive daily 60,000 lb of ripe tomatoes at 7 cents per pound from which it produces canned tomato juice, tomato sauce, and tomato paste. The canned products are packaged in 24-can cases. A can of juice uses 1 lb of fresh tomatoes, a can of sauce uses $\frac{1}{2}$ lb, and a can of paste uses $\frac{3}{4}$ lb. The company's daily share of the market is limited to 2000 cases of juice, 5000 cases of sauce, and 6000 cases of paste. The wholesale prices per case of juice, sauce, and paste are \$21, \$9, and \$12, respectively.
   (a) Develop an optimum daily production program for Popeye.
   (b) If the price per case for juice and paste remains fixed as given in the problem, use sensitivity analysis to determine the unit price range Popeye should charge for a case of sauce to keep the optimum product mix unchanged.
7. Dean's Furniture Company assembles regular and deluxe kitchen cabinets from precut lumber. The regular cabinets are painted white, and the deluxe are varnished. Both painting and varnishing are carried out in one department. The daily capacity of the assembly department is 200 regular cabinets and 150 deluxe. Varnishing a deluxe unit takes twice as much time as painting a regular one. If the painting/varnishing department is dedicated to the deluxe units only, it can complete 180 units daily. The company estimates that the revenues per unit for the regular and deluxe cabinets are \$100 and \$140, respectively.

(a) Formulate the problem as a linear program and find the optimal production schedule per day.

(b) Suppose that competition dictates that the price per unit of each of regular and deluxe cabinets be reduced to \$80. Use sensitivity analysis to determine whether or not the optimum solution in (a) remains unchanged.

8. *The 100% Optimality Rule.* A rule similar to the *100% feasibility rule* outlined in Problem 12, Set 3.6c, can also be developed for testing the effect of simultaneously changing all c_j to $c_j + d_j$, $j = 1, 2, \ldots, n$, on the optimality of the current solution. Suppose that $u_j \leq d_j \leq v_j$ is the optimality range obtained as a result of changing each c_j to $c_j + d_j$ one at a time, using the procedure in Section 3.6.3. In this case, $u_j \leq 0$ $(v_j \geq 0)$, because it represents the maximum allowable decrease (increase) in c_j that will keep the current solution optimal. For the cases where $u_j \leq d_j \leq v_j$, define r_j equal to $\frac{d_j}{v_j}$ if d_j is positive and $\frac{d_j}{u_j}$ if d_j is negative. By definition, $0 \leq r_j \leq 1$. The 100% rule says that a sufficient (but not necessary) condition for the current solution to remain optimal is that $r_1 + r_2 + \cdots + r_n \leq 1$. If the condition is not satisfied, the current solution may or may not remain optimal. The rule does not apply if d_j falls outside the specified ranges.

   Demonstrate that the 100% optimality rule is too weak to be consistently reliable as a decision-making tool by applying it to the following cases:

   (a) Parts (ii) and (iii) of Problem 1.

   (b) Part (b) of Problem 7.

3.6.4 Sensitivity Analysis with TORA, Solver, and AMPL

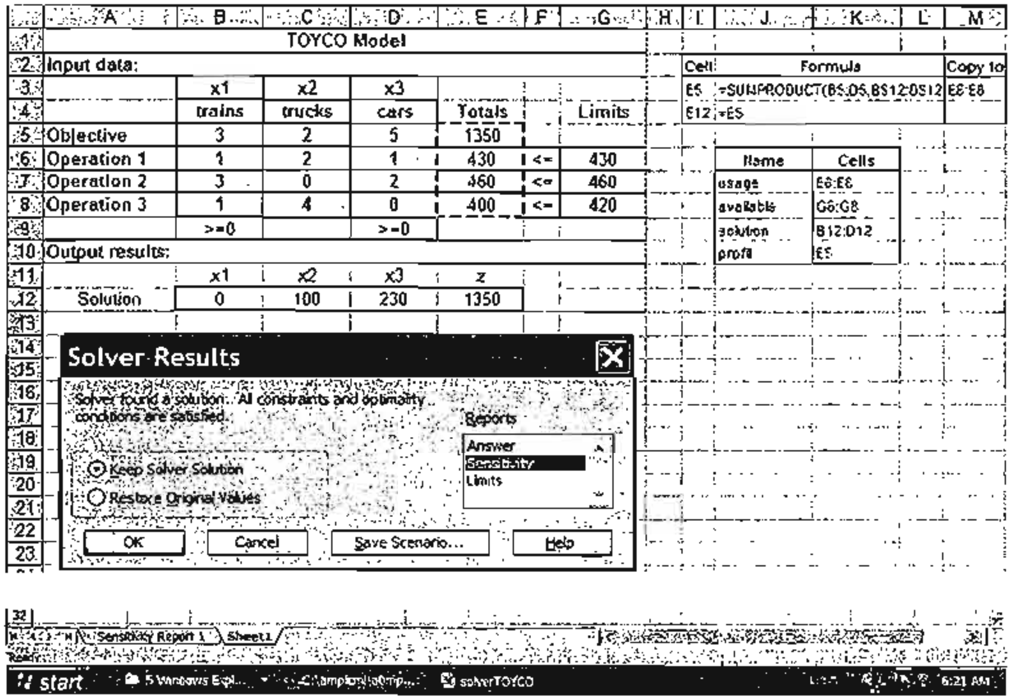
We now have all the tools needed to decipher the output provided by LP software, particularly with regard to sensitivity analysis. We will use the TOYCO example to demonstrate the TORA, Solver, and AMPL output.

TORA's LP output report provides the sensitivity analysis data automatically as shown in Figure 3.14 (file toraTOYCO.txt). The output includes the reduced costs and the dual prices as well as their allowable optimality and feasibility ranges.

FIGURE 3.14

TORA sensitivity analysis for the TOYCO model

Sensitivity Analysis				
Variable	CurrObjCoeff	MinObjCoeff	MaxObjCoeff	Reduced Cost
x1:	3.00	-infinity	7.00	4.00
x2:	2.00	0.00	10.00	0.00
x3:	5.00	2.33	infinity	0.00
Constraint	Curr RHS	Min RHS	Max RHS	Dual Price
1(<):	430.00	230.00	440.00	1.00
2(<):	460.00	440.00	860.00	2.00
3(<):	420.00	400.00	infinity	0.00



	Adjustable Cells						
	Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
9	$B$12	Solution x1	0	-4	3	4	1E+30
10	$C$12	Solution x2	100	0	2	8	2
11	$D$12	Solution x3	230	0	5	1E+30	2.666666667

	Constraints						
	Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
16	$E$6	Operation 1 Totals	430	1	430	10	200
17	$E$7	Operation 2 Totals	460	2	460	400	20
18	$E$8	Operation 3 Totals	400	0	420	1E+30	20

FIGURE 3.15

Excel Solver sensitivity analysis report for the TOYCO model

Figure 3.15 provides the Solver TOYCO model (file solverTOYCO.xls) and its sensitivity analysis report. After you click Solve in the **Solver Parameters** dialogue box, the new dialogue box **Solver Results** will give you the opportunity to request further details about the solution, including the important sensitivity analysis report. The report will be stored in a separate Excel sheet, as shown by the choices on the bottom of the screen. You can then click **Sensitivity Report 1** to view the results. The report is similar to TORA's with three exceptions: (1) The reduced cost carries an opposite sign. (2) The name *shadow price* replaces the name *dual price*. (3) The optimality ranges are for the changes d_j and D_i rather than for the total objective coefficients and constraints on the right-hand side. The differences are minor and the interpretation of the results remains the same.

In AMPL, the sensitivity analysis report is readily available. File amplTOYCO.txt provides the code necessary to determine the sensitivity analysis output. It requires the following additional statements:

```
option solver cplex;
option cplex_options 'sensitivity';
solve;
#-------------------------------sensitivity analysis
display oper.down,oper.current,oper.up,oper.dual>a.out;
display x.down,x.current,x.up,x.rc>a.out;
```

The CPLEX `option` statements are needed to be able to obtain the standard sensitivity analysis report. In the TOYCO model, the indexed variables and constraints use the root names `x` and `oper`, respectively. Using these names, the suggestive suffixes `.down`, `.current`, and `.up` in the `display` statements automatically generate the formatted sensitivity analysis report in Figure 3.16. The suffixes `.dual` and `.rc` provide the dual price and the reduced cost.

An alternative to AMPL's standard sensitivity analysis report is to actually solve the LP model for a range of values for the objective coefficients and the right-hand side of the constraints. AMPL automates this process through the use of `commands` (see Section A.7). Suppose in the TOYCO model, file amplTOYCO.txt, that we want to investigate the effect of making changes in `b[1]`, the total available time for operation 1. We can do so by moving `solve` and `display` from amplTOYCO.txt to a new file, which we arbitrarily name analysis.txt:

```
repeat while b[1]<=500
{
solve;
display z, x;
let b[1]:=b[1]+1;
};
```

Next, enter the following lines at the ampl prompt:

```
ampl: model amplTOYCO.txt;
ampl: commands analysis.txt;
```

```
: oper.down oper.current oper.up oper.dual    :=
1     230        430         440       1
2     440        460         860       2
3     400        420       1e+20       0
;

:      x.down      x.current      x.up   x.rc      :=
1    -1e+20              3          7     -4
2        0               2         10      0
3        2.33333         5       1e+20     0
;
```

FIGURE 3.16

AMPL sensitivity analysis report for the TOYCO model

The first line will provide the model and its data and the second line will provide the optimum solutions starting with `b[1]` at 430 (the initial value given in amplTOYCO.txt) and continuing in increments of 1 until `b[1]` reaches 500. An examination of the output will then allow us to study the sensitivity of the optimum solution to changes in `b[1]`. Similar procedures can be followed with other coefficients including the case of making simultaneous changes.

PROBLEM SET 3.6E[7]

1. Consider Problem 1, Set 2.3c (Chapter 2). Use the dual price to decide if it is worthwhile to increase the funding for year 4.
2. Consider Problem 2, Set 2.3c (Chapter 2).
   - (a) Use the dual prices to determine the overall return on investment.
   - (b) If you wish to spend $1000 on pleasure at the end of year 1, how would this affect the accumulated amount at the start of year 5?
3. Consider Problem 3, Set 2.3c (Chapter 2).
   - (a) Give an economic interpretation of the dual prices of the model.
   - (b) Show how the dual price associated with the upper bound on borrowed money at the beginning of the third quarter can be derived from the dual prices associated with the balance equations representing the in-out cash flow at the five designated dates of the year.
4. Consider Problem 4, Set 2.3c (Chapter 2). Use the dual prices to determine the rate of return associated with each year.

*5. Consider Problem 5, Set 2.3c (Chapter 2). Use the dual price to determine if it is worthwhile for the executive to invest more money in the plans.

6. Consider Problem 6, Set 2.3c (Chapter 2). Use the dual price to decide if it is advisable for the gambler to bet additional money.
7. Consider Problem 1, Set 2.3d (Chapter 2). Relate the dual prices to the unit production costs of the model.
8. Consider Problem 2, Set 2.3d (Chapter 2). Suppose that any additional capacity of machines 1 and 2 can be acquired only by using overtime. What is the maximum cost per hour the company should be willing to incur for either machine?

*9. Consider Problem 3, Set 2.3d (Chapter 2).
   - (a) Suppose that the manufacturer can purchase additional units of raw material A at $12 per unit. Would it be advisable to do so?
   - (b) Would you recommend that the manufacturer purchase additional units of raw material B at $5 per unit?

10. Consider Problem 10, Set 2.3e (Chapter 2).
    - (a) Which of the specification constraints impacts the optimum solution adversely?
    - (b) What is the most the company should pay per ton of each ore?

---

[7]Before answering the problems in this set, you are expected to generate the sensitivity analysis report using AMPL, Solver, or TORA.

REFERENCES

Bazaraa, M., J. Jarvis, and H. Sherali, *Linear Programming and Network Flows*, 2nd ed., Wiley, New York, 1990.

Dantzig, G., *Linear Programming and Extensions*, Princeton University Press, Princeton, NJ, 1963.

Dantzig, G., and M. Thapa, *Linear Programming 1: Introduction,* Springer, New York, 1997.

Fourer, R., D. Gay, and B. Kernighan, *AMPL, A Modeling Language for Mathematical Programming*, 2nd ed., Brooks/Cole-Thompson, Pacific Grove, CA, 2003.

Nering, E., and A. Tucker, *Linear Programming and Related Problems*, Academic Press, Boston, 1992.

Schrage, L., *Optimization Modeling with LINGO,* LINDO Systems, Inc., Chicago, 1999.

Taha, H., "Linear Programming," Chapter II-1 in *Handbook of Operations Research*, J. Moder and S. Elmaghraby (eds.), Van Nostrand Reinhold, New York, 1987.

CHAPTER 4

Duality and Post-Optimal Analysis

Chapter Guide. Chapter 3 dealt with the sensitivity of the optimal solution by determining the ranges for the model parameters that will keep the optimum basic solution unchanged. A natural sequel to sensitivity analysis is *post-optimal analysis*, where the goal is to determine the new optimum that results from making targeted changes in the model parameters. Although post-optimal analysis can be carried out using the simplex tableau computations in Section 3.6, this chapter is based entirely on the dual problem.

At a minimum, you will need to study the dual problem and its economic interpretation (Sections 4.1, 4.2, and 4.3). The mathematical definition of the dual problem in Section 4.1 is purely abstract. Yet, when you study Section 4.3, you will see that the dual problem leads to intriguing economic interpretations of the LP model, including *dual prices* and *reduced costs*. It also provides the foundation for the development of the new *dual simplex algorithm*, a prerequisite for post-optimal analysis. The dual simplex algorithm is also needed for integer programming in Chapter 9.

The *generalized simplex algorithm* in Section 4.4.2 is intended to show that the simplex method is not rigid, in the sense that you can modify the rules to handle problems that start both infeasible and nonoptimal. However, this material may be skipped without loss of continuity.

You may use TORA's interactive mode to reinforce your understanding of the computational details of the dual simplex method.

This chapter includes 14 solved examples, 56 end-of-section problems, and 2 cases. The cases are in Appendix E on the CD.

4.1 DEFINITION OF THE DUAL PROBLEM

The **dual** problem is an LP defined directly and systematically from the **primal** (or original) LP model. The two problems are so closely related that the optimal solution of one problem automatically provides the optimal solution to the other.

In most LP treatments, the dual is defined for various forms of the primal depending on the sense of optimization (maximization or minimization), types of constraints

($\leq$, $\geq$, or $=$), and orientation of the variables (nonnegative or unrestricted). This type of treatment is somewhat confusing, and for this reason we offer a *single* definition that automatically subsumes *all* forms of the primal.

Our definition of the dual problem requires expressing the primal problem in the *equation form* presented in Section 3.1 (all the constraints are equations with nonnegative right-hand side and all the variables are nonnegative). This requirement is consistent with the format of the simplex starting tableau. Hence, any results obtained from the primal optimal solution will apply directly to the associated dual problem.

To show how the dual problem is constructed, define the primal in *equation form* as follows:

$$\text{Maximize or minimize } z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i,\ i = 1, 2, \ldots, m$$

$$x_j \geq 0,\ j = 1, 2, \ldots, n$$

The variables x_j, $j = 1, 2, \ldots, n$, include the surplus, slack, and artificial variables, if any.

Table 4.1 shows how the dual problem is constructed from the primal. Effectively, we have

1. A dual variable is defined for each primal (constraint) equation.
2. A dual constraint is defined for each primal variable.
3. The constraint (column) coefficients of a primal variable define the left-hand-side coefficients of the dual constraint and its objective coefficient define the right-hand side.
4. The objective coefficients of the dual equal the right-hand side of the primal constraint equations.

TABLE 4.1 Construction of the Dual from the Primal

	Primal variables						
	x_1	x_2	$\cdots$	x_j	$\cdots$	x_n	
Dual variables	c_1	c_2	$\cdots$	c_j	$\cdots$	c_n	Right-hand side
y_1	a_{11}	a_{12}	$\cdots$	a_{1j}	$\cdots$	a_{1n}	b_1
y_2	a_{21}	a_{22}	$\cdots$	a_{2j}	$\cdots$	a_{2n}	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_m	a_{m1}	a_{m2}	$\cdots$	a_{mj}	$\cdots$	a_{mn}	b_m
				↑ *j*th dual constraint			↑ Dual objective coefficients

TABLE 4.2 Rules for Constructing the Dual Problem

Primal problem objective[a]	Dual problem		
	Objective	*Constraints type*	*Variables sign*
Maximization	Minimization	$\geq$	Unrestricted
Minimization	Maximization	$\leq$	Unrestricted

[a] All primal constraints are equations with nonnegative right-hand side and all the variables are nonnegative.

The rules for determining the sense of optimization (maximization or minimization), the type of the constraint ($\leq$, $\geq$, or $=$), and the sign of the dual variables are summarized in Table 4.2. Note that the sense of optimization in the dual is always opposite to that of the primal. An easy way to remember the constraint type in the dual (i.e., $\leq$ or $\geq$) is that if the dual objective is *minimization* (i.e., pointing *down*), then the constraints are all of the type $\geq$ (i.e., pointing *up*). The opposite is true when the dual objective is maximization.

The following examples demonstrate the use of the rules in Table 4.2 and also show that our definition incorporates all forms of the primal automatically.

Example 4.1-1

Primal	Primal in equation form	Dual variables
Maximize $z = 5x_1 + 12x_2 + 4x_3$	Maximize $z = 5x_1 + 12x_2 + 4x_3 + 0x_4$	
subject to	subject to	
$x_1 + 2x_2 + x_3 \leq 10$	$x_1 + 2x_2 + x_3 + x_4 = 10$	y_1
$2x_1 - x_2 + 3x_3 = 8$	$2x_1 - x_2 + 3x_3 + 0x_4 = 8$	y_2
$x_1, x_2, x_3 \geq 0$	$x_1, x_2, x_3, x_4 \geq 0$	

Dual Problem

$$\text{Minimize } w = 10y_1 + 8y_2$$

subject to

$$\begin{aligned} y_1 + 2y_2 &\geq 5 \\ 2y_1 - y_2 &\geq 12 \\ y_1 + 3y_2 &\geq 4 \\ \left.\begin{aligned} y_1 + 0y_2 &\geq 0 \\ y_1, y_2 \text{ unrestricted} \end{aligned}\right\} &\Rightarrow (y_1 \geq 0, y_2 \text{ unrestricted}) \end{aligned}$$

Example 4.1-2

Primal	Primal in equation form	Dual variables
Minimize $z = 15x_1 + 12x_2$	Minimize $z = 15x_1 + 12x_2 + 0x_3 + 0x_4$	
subject to	subject to	
$x_1 + 2x_2 \geq 3$	$x_1 + 2x_2 - x_3 + 0x_4 = 3$	y_1
$2x_1 - 4x_2 \leq 5$	$2x_1 - 4x_2 + 0x_3 + x_4 = 5$	y_2
$x_1, x_2 \geq 0$	$x_1, x_2, x_3, x_4 \geq 0$	

Dual Problem

$$\text{Maximize } w = 3y_1 + 5y_2$$

subject to

$$y_1 + 2y_2 \le 15$$
$$2y_1 - 4y_2 \le 12$$
$$\left.\begin{array}{r} -y_1 \qquad \le 0 \\ y_2 \le 0 \\ y_1, y_2 \text{ unrestricted} \end{array}\right\} \Rightarrow (y_1 \ge 0, y_2 \le 0)$$

Example 4.1-3

Primal	Primal in equation form	Dual variables
	Substitute $x_1 = x_1^+ - x_1^-$	
Maximize $z = 5x_1 + 6x_2$	Maximize $z = 5x_1^+ - 5x_1^- + 6x_2$	
subject to	subject to	
$x_1 + 2x_2 = 5$	$x_1^- - x_1^+ + 2x_2 = 5$	y_1
$-x_1 + 5x_2 \ge 3$	$-x_1^- + x_1^+ + 5x_2 - x_3 = 3$	y_2
$4x_1 + 7x_2 \le 8$	$4x_1^- - 4x_1^+ + 7x_2 + x_4 = 8$	y_3
x_1 unrestricted, $x_2 \ge 0$	$x_1^-, x_1^+, x_2, x_3, x_4 \ge 0$	

Dual Problem

$$\text{Minimize } z = 5y_1 + 3y_2 + 8y_3$$

subject to

$$\left.\begin{array}{r} y_1 - y_2 + 4y_3 \ge 5 \\ -y_1 + y_2 - 4y_3 \ge -5 \end{array}\right\} \Rightarrow (y_1 - y_2 + 4y_3 = 5)$$

$$2y_1 + 5y_2 + 7y_3 \ge 6$$

$$\left.\begin{array}{r} -y_2 \qquad \ge 0 \\ y_3 \ge 0 \\ y_1, y_2, y_3 \text{ unrestricted} \end{array}\right\} \Rightarrow (y_1 \text{ unrestricted}, y_2 \le 0, y_3 \ge 0)$$

The first and second constraints are replaced by an equation. The general rule in this case is that an unrestricted primal variable always corresponds to an equality dual constraint. Conversely, a primal equation produces an unrestricted dual variable, as the first primal constraint demonstrates.

Summary of the Rules for Constructing the Dual. The general conclusion from the preceding examples is that the variables and constraints in the primal and dual problems are defined by the rules in Table 4.3. It is a good exercise to verify that these explicit rules are subsumed by the general rules in Table 4.2.

TABLE 4.3 Rules for Constructing the Dual Problem

Maximization problem		Minimization problem
Constraints		*Variables*
$\geq$	$\Leftrightarrow$	≤ 0
$\leq$	$\Leftrightarrow$	≥ 0
$=$	$\Leftrightarrow$	Unrestricted
Variables		*Constraints*
≥ 0	$\Leftrightarrow$	$\geq$
≤ 0	$\Leftrightarrow$	$\leq$
Unrestricted	$\Leftrightarrow$	$=$

Note that the table does not use the designation primal and dual. What matters here is the sense of optimization. If the primal is maximization, then the dual is minimization, and vice versa.

PROBLEM SET 4.1A

1. In Example 4.1-1, derive the associated dual problem if the sense of optimization in the primal problem is changed to minimization.

*2. In Example 4.1-2, derive the associated dual problem given that the primal problem is augmented with a third constraint, $3x_1 + x_2 = 4$.

3. In Example 4.1-3, show that even if the sense of optimization in the primal is changed to minimization, an unrestricted primal variable always corresponds to an equality dual constraint.

4. Write the dual for each of the following primal problems:

(a) Maximize $z = -5x_1 + 2x_2$
subject to

$$-x_1 + x_2 \leq -2$$
$$2x_1 + 3x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

(b) Minimize $z = 6x_1 + 3x_2$
subject to

$$6x_1 - 3x_2 + x_3 \geq 2$$
$$3x_1 + 4x_2 + x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0$$

***(c)** Maximize $z = x_1 + x_2$
subject to

$$2x_1 + x_2 = 5$$
$$3x_1 - x_2 = 6$$
$$x_1, x_2 \text{ unrestricted}$$

*5. Consider Example 4.1-1. The application of the simplex method to the primal requires the use of an artificial variable in the second constraint of the standard primal to secure a starting basic solution. Show that the presence of an artificial primal in equation form variable does not affect the definition of the dual because it leads to a redundant dual constraint.

6. True or False?
   (a) The dual of the dual problem yields the original primal.
   (b) If the primal constraint is originally in equation form, the corresponding dual variable is necessarily unrestricted.
   (c) If the primal constraint is of the type $\leq$, the corresponding dual variable will be nonnegative (nonpositive) if the primal objective is maximization (minimization).
   (d) If the primal constraint is of the type $\geq$, the corresponding dual variable will be nonnegative (nonpositive) if the primal objective is minimization (maximization).
   (e) An unrestricted primal variable will result in an equality dual constraint.

4.2 PRIMAL-DUAL RELATIONSHIPS

Changes made in the original LP model will change the elements of the current optimal tableau, which in turn may affect the optimality and/or the feasibility of the current solution. This section introduces a number of primal-dual relationships that can be used to recompute the elements of the optimal simplex tableau. These relationships will form the basis for the economic interpretation of the LP model as well as for post-optimality analysis.

This section starts with a brief review of matrices, a convenient tool for carrying out the simplex tableau computations.

4.2.1 Review of Simple Matrix Operations

The simplex tableau computations use only three elementary matrix operations: (row vector) $\times$ (matrix), (matrix) $\times$ (column vector), and (scalar) $\times$ (matrix). These operations are summarized here for convenience. First, we introduce some matrix definitions:[1]

1. A *matrix*, $\mathbf{A}$, of size $(m \times n)$ is a rectangular array of elements with m rows and n columns.
2. A *row vector*, $\mathbf{V}$, of size m is a $(1 \times m)$ matrix.
3. A *column vector*, $\mathbf{P}$, of size n is an $(n \times 1)$ matrix.

These definitions can be represented mathematically as

$$\mathbf{V} = (v_1, v_2, \ldots, v_m), \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \vdots & a_{1n} \\ a_{21} & a_{22} & \vdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \vdots & a_{mn} \end{pmatrix}, \mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ \cdots \\ p_n \end{pmatrix}$$

[1]Appendix D on the CD provides a more complete review of matrices.

1. (Row vector $\times$ matrix, VA). The operation is defined only if the size of the row vector **V** equals the number of rows of **A**. In this case,

$$\mathbf{VA} = \left(\sum_{i=1}^{m} v_i a_{i1}, \sum_{i=1}^{m} v_i a_{i2}, \ldots, \sum_{i=1}^{m} v_i a_{in}\right)$$

For example,

$$(11, 22, 33)\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = (1 \times 11 + 3 \times 22 + 5 \times 33, 2 \times 11 + 4 \times 22 + 6 \times 33)$$

$$= (242, 308)$$

2. (Matrix $\times$ column vector, AP). The operation is defined only if the number of columns of **A** equals the size of column vector **P**. In this case,

$$\mathbf{AP} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} p_j \\ \sum_{j=1}^{n} a_{2j} p_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} p_j \end{pmatrix}$$

As an illustration, we have

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}\begin{pmatrix} 11 \\ 22 \\ 33 \end{pmatrix} = \begin{pmatrix} 1 \times 11 + 3 \times 22 + 5 \times 33 \\ 2 \times 11 + 4 \times 22 + 6 \times 33 \end{pmatrix} = \begin{pmatrix} 242 \\ 308 \end{pmatrix}$$

3. (Scalar $\times$ matrix, αA). Given the scalar (or constant) quantity α, the multiplication operation $\alpha\mathbf{A}$ will result in a matrix of the same size as **A** whose (i, j)th element equals αa_{ij}. For example, given $\alpha = 10$,

$$(10)\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \end{pmatrix}$$

In general, $\alpha\mathbf{A} = \mathbf{A}\alpha$. The same operation is extended equally to the multiplication of vectors by scalars. For example, $\alpha\mathbf{V} = \mathbf{V}\alpha$ and $\alpha\mathbf{P} = \mathbf{P}\alpha$.

PROBLEM SET 4.2A

1. Consider the following matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \mathbf{P}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\mathbf{V}_1 = (11, 22), \mathbf{V}_2 = (-1, -2, -3)$$

In each of the following cases, indicate whether the given matrix operation is legitimate, and, if so, calculate the result.

*(a) $\mathbf{AV}_1$
(b) $\mathbf{AP}_1$
(c) $\mathbf{AP}_2$
(d) $\mathbf{V}_1\mathbf{A}$
*(e) $\mathbf{V}_2\mathbf{A}$
(f) $\mathbf{P}_1\mathbf{P}_2$
(g) $\mathbf{V}_1\mathbf{P}_1$

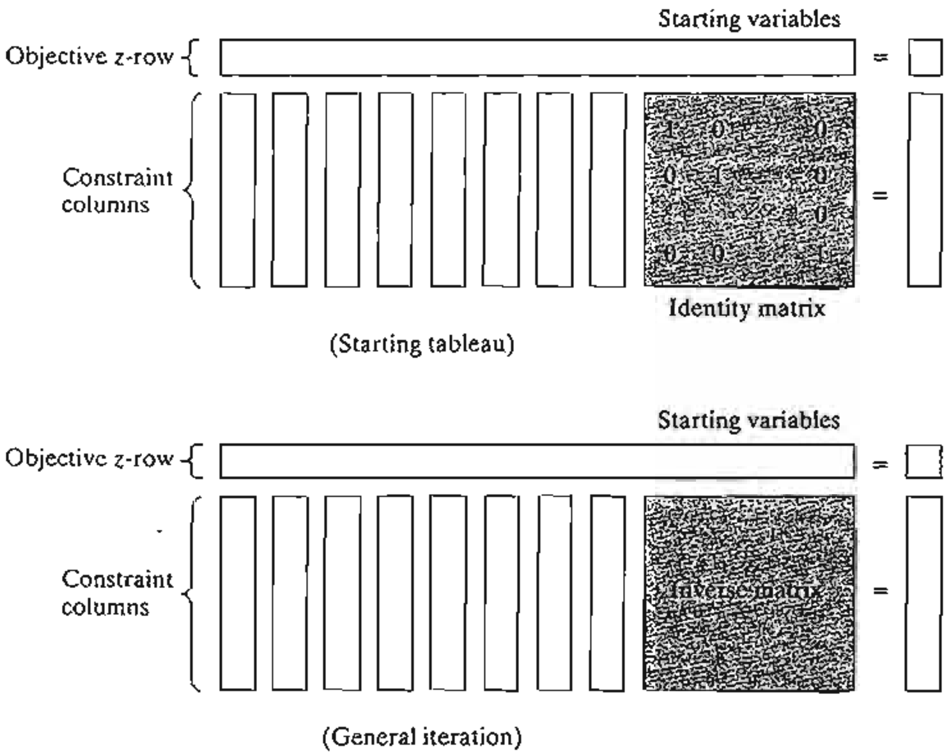
4.2.2 Simplex Tableau Layout

In Chapter 3, we followed a specific format for setting up the simplex tableau. This format is the basis for the development in this chapter.

Figure 4.1 gives a schematic representation of the *starting* and *general* simplex tableaus. In the starting tableau, the constraint coefficients under the starting variables form an **identity matrix** (all main-diagonal elements equal 1 and all off-diagonal elements equal zero). With this arrangement, subsequent iterations of the simplex tableau generated by the Gauss-Jordan row operations (see Chapter 3) will modify the elements of the identity matrix to produce what is known as the **inverse matrix**. As we will see in the remainder of this chapter, the inverse matrix is key to computing all the elements of the associated simplex tableau.

FIGURE 4.1

Schematic representation of the starting and general simplex tableaus



PROBLEM SET 4.2B

1. Consider the optimal tableau of Example 3.3-1.
   *(a) Identify the optimal inverse matrix.
   (b) Show that the right-hand side equals the inverse multiplied by the original right-hand side vector of the original constraints.
2. Repeat Problem 1 for the last tableau of Example 3.4-1.

4.2.3 Optimal Dual Solution

The primal and dual solutions are so closely related that the optimal solution of either problem directly yields (with little additional computation) the optimal solution to the other. Thus, in an LP model in which the number of variables is considerably smaller than the number of constraints, computational savings may be realized by solving the dual, from which the primal solution is determined automatically. This result follows because the amount of simplex computation depends largely (though not totally) on the number of constraints (see Problem 2, Set 4.2c).

This section provides two methods for determining the dual values. Note that the dual of the dual is itself the primal, which means that the dual solution can also be used to yield the optimal primal solution automatically.

Method 1.

$$\begin{pmatrix}\text{Optimal value of}\\ \textit{dual}\text{ variable } y_i\end{pmatrix} = \begin{pmatrix}\text{Optimal primal } z\text{-coefficient of } \textit{starting} \text{ variable } x_i\\ +\\ \textit{Original} \text{ objective coefficient of } x_i\end{pmatrix}$$

Method 2.

$$\begin{pmatrix}\text{Optimal values}\\ \text{of } \textit{dual} \text{ variables}\end{pmatrix} = \begin{pmatrix}\text{Row vector of}\\ \text{original objective coefficients}\\ \text{of optimal } \textit{primal} \text{ basic variables}\end{pmatrix} \times \begin{pmatrix}\text{Optimal } \textit{primal}\\ \text{inverse}\end{pmatrix}$$

The elements of the row vector must appear in the same order in which the basic variables are listed in the *Basic* column of the simplex tableau.

Example 4.2-1

Consider the following LP:

$$\text{Maximize } z = 5x_1 + 12x_2 + 4x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 10\\ 2x_1 - x_2 + 3x_3 &= 8\\ x_1, x_2, x_3 &\geq 0\end{aligned}$$

To prepare the problem for solution by the simplex method, we add a slack x_4 in the first constraint and an artificial R in the second. The resulting primal and the associated dual problems are thus defined as follows:

Primal	Dual
Maximize $z = 5x_1 + 12x_2 + 4x_3 - MR$	Minimize $w = 10y_1 + 8y_2$
subject to	subject to
$x_1 + 2x_2 + x_3 + x_4 = 10$	$y_1 + 2y_2 \geq 5$
$2x_1 - x_2 + 3x_3 + R = 8$	$2y_1 - y_2 \geq 12$
$x_1, x_2, x_3, x_4, R \geq 0$	$y_1 + 3y_2 \geq 4$
	$y_1 \geq 0$
	$y_2 \geq -M$ ($\Rightarrow y_2$ unrestricted)

Table 4.4 provides the optimal primal tableau.

We now show how the optimal dual values are determined using the two methods described at the start of this section.

Method 1. In Table 4.4, the starting primal variables x_4 and R uniquely correspond to the dual variables y_1 and y_2, respectively. Thus, we determine the optimum dual solution as follows:

Starting primal basic variables	x_4	R
z-equation coefficients	$\frac{29}{5}$	$-\frac{2}{5} + M$
Original objective coefficient	0	$-M$
Dual variables	y_1	y_2
Optimal dual values	$\frac{29}{5} + 0 = \frac{29}{5}$	$-\frac{2}{5} + M + (-M) = -\frac{2}{5}$

Method 2. The optimal inverse matrix, highlighted under the starting variables x_4 and R, is given in Table 4.4 as

$$\text{Optimal inverse} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

First, we note that the optimal primal variables are listed in the tableau in *row order* as x_2 and then x_1. This means that the elements of the original objective coefficients for the two variables must appear in the same order—namely,

$$(\text{Original objective coefficients}) = (\text{Coefficient of } x_2, \text{coefficient of } x_1)$$
$$= (12, 5)$$

TABLE 4.4 Optimal Tableau of the Primal of Example 4.2-1

Basic	x_1	x_2	x_3	x_4	R	Solution
z	0	0	$\frac{3}{5}$	$\frac{29}{5}$	$-\frac{2}{5} + M$	$54\frac{4}{5}$
x_2	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{12}{5}$
x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{26}{5}$

Thus, the optimal dual values are computed as

$$(y_1, y_2) = \begin{pmatrix}\text{Original objective}\\ \text{coefficients of } x_2, x_1\end{pmatrix} \times (\text{Optimal inverse})$$

$$= (12, 5)\begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}$$

$$= \left(\tfrac{29}{5}, -\tfrac{2}{5}\right)$$

Primal-dual objective values. Having shown how the optimal dual values are determined, next we present the relationship between the primal and dual objective values. For any pair of *feasible* primal and dual solutions,

$$\begin{pmatrix}\text{Objective value in the}\\ \textit{maximization} \text{ problem}\end{pmatrix} \leq \begin{pmatrix}\text{Objective value in the}\\ \textit{minimization} \text{ problem}\end{pmatrix}$$

At the optimum, the relationship holds as a strict equation. The relationship does not specify which problem is primal and which is dual. Only the sense of optimization (maximization or minimization) is important in this case.

The optimum cannot occur with z strictly less than w (i.e., $z < w$) because, no matter how close z and w are, there is always room for improvement, which contradicts optimality as Figure 4.2 demonstrates.

Example 4.2-2

In Example 4.2-1, $\left(x_1 = 0, x_2 = 0, x_3 = \frac{8}{3}\right)$ and $(y_1 = 6, y_2 = 0)$ are feasible primal and dual solutions. The associated values of the objective functions are

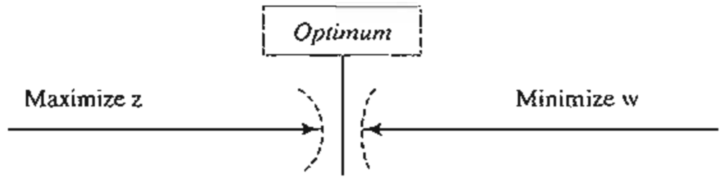
$$z = 5x_1 + 12x_2 + 4x_3 = 5(0) + 12(0) + 4\left(\tfrac{8}{3}\right) = 10\tfrac{2}{3}$$

$$w = 10y_1 + 8y_2 = 10(6) + 8(0) = 60$$

Thus, $z \left(= 10\frac{2}{3}\right)$ for the maximization problem (primal) is less than w (= 60) for the minimization problem (dual). The optimum value of $z \left(= 54\frac{4}{5}\right)$ falls within the range $\left(10\frac{2}{3}, 60\right)$.

FIGURE 4.2

Relationship between maximum z and minimum w



PROBLEM SET 4.2C

1. Find the optimal value of the objective function for the following problem by inspecting only its dual. (Do not solve the dual by the simplex method.)

$$\text{Minimize } z = 10x_1 + 4x_2 + 5x_3$$

subject to

$$5x_1 - 7x_2 + 3x_3 \geq 50$$
$$x_1, x_2, x_3 \geq 0$$

2. Solve the dual of the following problem, then find its optimal solution from the solution of the dual. Does the solution of the dual offer computational advantages over solving the primal directly?

$$\text{Minimize } z = 5x_1 + 6x_2 + 3x_3$$

subject to

$$\begin{aligned} 5x_1 + 5x_2 + 3x_3 &\geq 50 \\ x_1 + x_2 - x_3 &\geq 20 \\ 7x_1 + 6x_2 - 9x_3 &\geq 30 \\ 5x_1 + 5x_2 + 5x_3 &\geq 35 \\ 2x_1 + 4x_2 - 15x_3 &\geq 10 \\ 12x_1 + 10x_2 \qquad &\geq 90 \\ x_2 - 10x_3 &\geq 20 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

*3. Consider the following LP:

$$\text{Maximize } z = 5x_1 + 2x_2 + 3x_3$$

subject to

$$x_1 + 5x_2 + 2x_3 = 30$$
$$x_1 - 5x_2 - 6x_3 \leq 40$$
$$x_1, x_2, x_3 \geq 0$$

Given that the artificial variable x_4 and the slack variable x_5 form the starting basic variables and that M was set equal to 100 when solving the problem, the *optimal* tableau is given as

Basic	x_1	x_2	x_3	x_4	x_5	Solution
z	0	23	7	105	0	150
x_1	1	5	2	1	0	30
x_5	0	−10	−8	−1	1	10

Write the associated dual problem and determine its optimal solution in two ways.

4. Consider the following LP:

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

The starting solution consists of artificial x_4 and x_5 for the first and second constraints and slack x_6 for the third constraint. Using $M = 100$ for the artificial variables, the optimal tableau is given as

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	0	−98.6	−100	−.2	3.4
x_1	1	0	0	.4	0	−.2	.4
x_2	0	1	0	.2	0	.6	1.8
x_3	0	0	1	1	−1	1	1.0

Write the associated dual problem and determine its optimal solution in two ways.

5. Consider the following LP:

$$\text{Maximize } z = 2x_1 + 4x_2 + 4x_3 - 3x_4$$

subject to

$$x_1 + x_2 + x_3 = 4$$
$$x_1 + 4x_2 + x_4 = 8$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Using x_3 and x_4 as starting variables, the optimal tableau is given as

Basic	x_1	x_2	x_3	x_4	Solution
z	2	0	0	3	16
x_3	.75	0	1	−.25	2
x_2	.25	1	0	.25	2

Write the associated dual problem and determine its optimal solution in two ways.

***6.** Consider the following LP:

$$\text{Maximize } z = x_1 + 5x_2 + 3x_3$$

subject to

$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 - x_2 = 4$$
$$x_1, x_2, x_3 \geq 0$$

The starting solution consists of x_3 in the first constraint and an artificial x_4 in the second constraint with $M = 100$. The optimal tableau is given as

Basic	x_1	x_2	x_3	x_4	Solution
z	0	2	0	99	5
x_3	1	2.5	1	−.5	1
x_1	0	−.5	0	.5	2

Write the associated dual problem and determine its optimal solution in two ways.

7. Consider the following set of inequalities:

$$\begin{aligned} 2x_1 + 3x_2 &\le 12 \\ -3x_1 + 2x_2 &\le -4 \\ 3x_1 - 5x_2 &\le 2 \\ x_1 &\text{ unrestricted} \\ x_2 &\ge 0 \end{aligned}$$

A feasible solution can be found by augmenting the trivial objective function Maximize $z = x_1 + x_2$ and then solving the problem. Another way is to solve the dual; from which a solution for the set of inequalities can be found. Apply the two methods.

8. Estimate a range for the optimal objective value for the following LPs:

*(a) Minimize $z = 5x_1 + 2x_2$
subject to

$$\begin{aligned} x_1 - x_2 &\ge 3 \\ 2x_1 + 3x_2 &\ge 5 \\ x_1, x_2 &\ge 0 \end{aligned}$$

(b) Maximize $z = x_1 + 5x_2 + 3x_3$
subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 - x_2 &= 4 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

(c) Maximize $z = 2x_1 + x_2$
subject to

$$\begin{aligned} x_1 - x_2 &\le 10 \\ 2x_1 &\le 40 \\ x_1, x_2 &\ge 0 \end{aligned}$$

(d) Maximize $z = 3x_1 + 2x_2$
subject to

$$\begin{aligned} 2x_1 + x_2 &\le 3 \\ 3x_1 + 4x_2 &\le 12 \\ x_1, x_2 &\ge 0 \end{aligned}$$

9. In Problem 7(a), let y_1 and y_2 be the dual variables. Determine whether the following pairs of primal-dual solutions are optimal:

*(a) $(x_1 = 3, x_2 = 1; y_1 = 4, y_2 = 1)$

(b) $(x_1 = 4, x_2 = 1; y_1 = 1, y_2 = 0)$

(c) $(x_1 = 3, x_2 = 0; y_1 = 5, y_2 = 0)$

4.2.4 Simplex Tableau Computations

This section shows how *any iteration* of the entire simplex tableau can be generated from the *original* data of the problem, the *inverse* associated with the iteration, and the dual problem. Using the layout of the simplex tableau in Figure 4.1, we can divide the computations into two types:

1. Constraint columns (left- and right-hand sides).
2. Objective z-row.

Formula 1: Constraint Column Computations. In any simplex iteration, a left-hand or a right-hand side column is computed as follows:

$$\begin{pmatrix}\text{Constraint column}\\ \text{in iteration } i\end{pmatrix} = \begin{pmatrix}\text{Inverse in}\\ \text{iteration } i\end{pmatrix} \times \begin{pmatrix}\text{Original}\\ \text{constraint column}\end{pmatrix}$$

Formula 2: Objective z-row Computations. In any simplex iteration, the objective equation coefficient (reduced cost) of x_j is computed as follows:

$$\begin{pmatrix}\textit{Primal } z\text{-equation}\\ \text{coefficient of variable } x_j\end{pmatrix} = \begin{pmatrix}\text{Left-hand side of}\\ j\text{th } \textit{dual} \text{ constraint}\end{pmatrix} - \begin{pmatrix}\text{Right-hand side of}\\ j\text{th } \textit{dual} \text{ constraint}\end{pmatrix}$$

Example 4.2-3

We use the LP in Example 4.2-1 to illustrate the application of Formulas 1 and 2. From the optimal tableau in Table 4.4, we have

$$\text{Optimal inverse} = \begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}$$

The use of Formula 1 is illustrated by computing all the left- and right-hand side columns of the optimal tableau:

$$\begin{pmatrix}x_1\text{-column in}\\ \text{optimal iteration}\end{pmatrix} = \begin{pmatrix}\text{Inverse in}\\ \text{optimal iteration}\end{pmatrix} \times \begin{pmatrix}\text{original}\\ x_1\text{-column}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix} \times \begin{pmatrix}1\\ 2\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$$

In a similar manner, we compute the remaining constraint columns; namely,

$$\begin{pmatrix} x_2\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_3\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \\ \frac{7}{5} \end{pmatrix}$$

$$\begin{pmatrix} x_4\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ \frac{1}{5} \end{pmatrix}$$

$$\begin{pmatrix} R\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{pmatrix}$$

$$\begin{pmatrix} \text{Right-hand side} \\ \text{column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 10 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{12}{5} \\ \frac{26}{5} \end{pmatrix}$$

Next, we demonstrate how the objective row computations are carried out using Formula 2. The optimal values of the dual variables, $(y_1, y_2) = \left(\frac{29}{5}, -\frac{2}{5}\right)$, were computed in Example 4.2-1 using two different methods. These values are used in Formula 2 to determine the associated z-coefficients; namely,

$$\begin{aligned}
z\text{-cofficient of } x_1 &= y_1 + 2y_2 - 5 &&= \tfrac{29}{5} + 2 \times -\tfrac{2}{5} - 5 &&= 0 \\
z\text{-cofficient of } x_2 &= 2y_1 - y_2 - 12 &&= 2 \times \tfrac{29}{5} - \left(-\tfrac{2}{5}\right) - 12 &&= 0 \\
z\text{-cofficient of } x_3 &= y_1 + 3y_2 - 4 &&= \tfrac{29}{5} + 3 \times -\tfrac{2}{5} - 4 &&= \tfrac{3}{5} \\
z\text{-cofficient of } x_4 &= y_1 - 0 &&= \tfrac{29}{5} - 0 &&= \tfrac{29}{5} \\
z\text{-cofficient of } R &= y_2 - (-M) &&= -\tfrac{2}{5} - (-M) &&= -\tfrac{2}{5} + M
\end{aligned}$$

Notice that Formula 1 and Formula 2 calculations can be applied at any iteration of either the primal or the dual problems. All we need is the inverse associated with the (primal or dual) iteration and the original LP data.

PROBLEM SET 4.2D

1. Generate the first simplex iteration of Example 4.2-1 (you may use TORA's Iterations $\Rightarrow$ M-method for convenience), then use Formulas 1 and 2 to verify all the elements of the resulting tableau.
2. Consider the following LP model:

$$\text{Maximize } z = 4x_1 + 14x_2$$

subject to

$$\begin{aligned}
2x_1 + 7x_2 + x_3 \quad &= 21 \\
7x_1 + 2x_2 \quad + x_4 &= 21 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}$$

Check the optimality and feasibility of each of the following basic solutions.

***(a)** Basic variables = (x_2, x_4), Inverse = $\begin{pmatrix} \frac{1}{7} & 0 \\ -\frac{2}{7} & 1 \end{pmatrix}$

(b) Basic variables = (x_2, x_3), Inverse = $\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{7}{2} \end{pmatrix}$

(c) Basic variables = (x_2, x_1), Inverse = $\begin{pmatrix} \frac{7}{45} & -\frac{2}{45} \\ -\frac{2}{45} & \frac{7}{45} \end{pmatrix}$

(d) Basic variables = (x_1, x_4), Inverse = $\begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{7}{2} & 1 \end{pmatrix}$

3. Consider the following LP model:

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 \qquad &= 30 \\ 3x_1 \qquad + 2x_3 \qquad + x_5 \qquad &= 60 \\ x_1 + 4x_2 \qquad\qquad\qquad + x_6 &= 20 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

Check the optimality and feasibility of the following basic solutions:

(a) Basic variables = (x_4, x_3, x_6), Inverse = $\begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) Basic variables = (x_2, x_3, x_1), Inverse = $\begin{pmatrix} \frac{1}{4} & -\frac{1}{8} & \frac{1}{8} \\ \frac{3}{2} & -\frac{1}{4} & -\frac{3}{4} \\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

(c) Basic variables = (x_2, x_3, x_6), Inverse = $\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix}$

***4.** Consider the following LP model:

$$\text{Minimize } z = 2x_1 + x_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 - x_3 \qquad &= 3 \\ 4x_1 + 3x_2 \qquad - x_4 \qquad &= 6 \\ x_1 + 2x_2 \qquad\qquad + x_5 &= 3 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Compute the entire simplex tableau associated with the following basic solution and check it for optimality and feasibility.

$$\text{Basic variables} = (x_1, x_2, x_5), \text{ Inverse} = \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

5. Consider the following LP model:

$$\text{Maximize } z = 5x_1 + 12x_2 + 4x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 10 \\ 2x_1 - x_2 + 3x_3 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(a) Identify the best solution from among the following basic feasible solutions:

(i) $\text{Basic variables} = (x_4, x_3), \text{ Inverse} = \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$

(ii) $\text{Basic variables} = (x_2, x_1), \text{ Inverse} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$

(iii) $\text{Basic variables} = (x_2, x_3), \text{ Inverse} = \begin{pmatrix} \frac{3}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{pmatrix}$

(b) Is the solution obtained in (a) optimum for the LP model?

6. Consider the following LP model:

$$\text{Maximize } z = 5x_1 + 2x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 + 5x_2 + 2x_3 &\leq b_1 \\ x_1 - 5x_2 - 6x_3 &\leq b_2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The following optimal tableau corresponds to specific values of b_1 and b_2:

Basic	x_1	x_2	x_3	x_4	x_5	Solution
z	0	a	7	d	e	150
x_1	1	b	2	1	0	30
x_5	0	c	−8	−1	1	10

Determine the following:

(a) The right-hand-side values, b_1 and b_2.

(b) The optimal dual solution.

(c) The elements a, b, c, d, e.

*7. The following is the optimal tableau for a maximization LP model with three ($\leq$) constraints and all nonnegative variables. The variables x_3, x_4, and x_5 are the slacks associated with the three constraints. Determine the associated optimal objective value in two different ways by using the primal and dual objective functions.

Basic	x_1	x_2	x_3	x_4	x_5	Solution
z	0	0	0	3	2	?
x_3	0	0	1	1	−1	2
x_2	0	1	0	1	0	6
x_1	1	0	0	−1	1	2

8. Consider the following LP:

$$\text{Maximize } z = 2x_1 + 4x_2 + 4x_3 - 3x_4$$

subject to

$$x_1 + x_2 + x_3 = 4$$
$$x_1 + 4x_2 + x_4 = 8$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Use the dual problem to show that the basic solution (x_1, x_2) is not optimal.

9. Show that Method 1 in Section 4.2.3 for determining the optimal dual values is actually based on the Formula 2 in Section 4.2.4.

4.3 ECONOMIC INTERPRETATION OF DUALITY

The linear programming problem can be viewed as a resource allocation model in which the objective is to maximize revenue subject to the availability of limited resources. Looking at the problem from this standpoint, the associated dual problem offers interesting economic interpretations of the LP resource allocation model.

To formalize the discussion, we consider the following representation of the general primal and dual problems:

Primal	Dual
Maximize $z = \sum_{j=1}^{n} c_j x_j$	Minimize $w = \sum_{i=1}^{m} b_i y_i$
subject to	subject to
$\sum_{j=1}^{n} a_{ij} x_j \leq b_i,\ i = 1, 2, \ldots, m$	$\sum_{i=1}^{m} a_{ij} y_i \geq c_j,\ j = 1, 2, \ldots, n$
$x_j \geq 0,\ j=1, 2, \ldots, n$	$y_i \geq 0,\ i = 1, 2, \ldots, m$

Viewed as a resource allocation model, the primal problem has n economic activities and m resources. The coefficient c_j in the primal represents the revenue per unit of activity j. Resource i, whose maximum availability is b_i, is consumed at the rate a_{ij} units per unit of activity j.

4.3.1 Economic Interpretation of Dual Variables

Section 4.2.3 states that for any two primal and dual *feasible* solutions, the values of the objective functions, when finite, must satisfy the following inequality:

$$z = \sum_{j=1}^{n} c_j x_j \le \sum_{i=1}^{m} b_i y_i = w$$

The strict equality, $z = w$, holds when both the primal and dual solutions are optimal.

Let us consider the optimal condition $z = w$ first. Given that the primal problem represents a resource allocation model, we can think of z as representing revenue dollars. Because b_i represents the number of units available of resource i, the equation $z = w$ can be expressed dimensionally as

$$\$ = \sum_i (\text{units of resource } i) \times (\$ \text{ per unit of resource } i)$$

This means that the dual variable, y_i, represents the **worth per unit** of resource i. As stated in Section 3.6, the standard name **dual** (or **shadow**) **price** of resource i replaces the name *worth per unit* in all LP literature and software packages.

Using the same logic, the inequality $z < w$ associated with any two feasible primal and dual solutions is interpreted as

$$(\text{Revenue}) < (\text{Worth of resources})$$

This relationship says that so long as the total revenue from all the activities is less than the worth of the resources, the corresponding primal and dual solutions are not optimal. Optimality (maximum revenue) is reached only when the resources have been exploited completely, which can happen only when the input (worth of the resources) equals the output (revenue dollars). In economic terms, the system is said to be *unstable* (nonoptimal) when the input (worth of the resources) exceeds the output (revenue). Stability occurs only when the two quantities are equal.

Example 4.3-1

The Reddy Mikks model (Example 2.1-1) and its dual are given as:

Reddy Mikks primal	Reddy Mikks dual
Maximize $z = 5x_1 + 4x_2$	Minimize $w = 24y_1 + 6y_2 + y_3 + 2y_4$
subject to	subject to
$6x_1 + 4x_2 \le 24$ (resource 1, $M1$)	$6y_1 + y_2 - y_3 \ge 5$
$x_1 + 2x_2 \le 6$ (resource 2, $M2$)	$4y_1 + 2y_2 + y_3 + y_4 \ge 4$
$-x_1 + x_2 \le 1$ (resource 3, market)	$y_1, y_2, y_3, y_4 \ge 0$
$x_2 \le 2$ (resource 4, demand)	
$x_1, x_2 \ge 0$	
Optimal solution:	Optimal solution:
$x_1 = 3, x_2 = 1.5, z = 21$	$y_1 = .75, y_2 = 0.5, y_3 = y_4 = 0, w = 21$

Briefly, the Reddy Mikks model deals with the production of two types of paint (interior and exterior) using two raw materials $M1$ and $M2$ (resources 1 and 2) and subject to market and demand limits represented by the third and fourth constraints. The model determines the amounts (in tons/day) of interior and exterior paints that maximize the daily revenue (expressed in thousands of dollars).

The optimal dual solution shows that the dual price (worth per unit) of raw material $M1$ (resource 1) is $y_1 = .75$ (or \$750 per ton), and that of raw material $M2$ (resource 2) is $y_2 = .5$ (or \$500 per ton). These results hold true for specific *feasibility ranges* as we showed in Section 3.6. For resources 3 and 4, representing the market and demand limits, the dual prices are both zero, which indicates that their associated resources are abundant. Hence, their worth per unit is zero.

PROBLEM SET 4.3A

1. In Example 4.3-1, compute the change in the optimal revenue in each of the following cases (use TORA output to obtain the *feasibility ranges*):
   (a) The constraint for raw material $M1$ (resource 1) is $6x_1 + 4x_2 \leq 22$.
   (b) The constraint for raw material $M2$ (resource 2) is $x_1 + 2x_2 \leq 4.5$.
   (c) The market condition represented by resource 4 is $x_2 \leq 10$.

*2. NWAC Electronics manufactures four types of simple cables for a defense contractor. Each cable must go through four sequential operations: splicing, soldering, sleeving, and inspection. The following table gives the pertinent data of the situation.

	Minutes per unit				
Cable	*Splicing*	*Soldering*	*Sleeving*	*Inspection*	Unit revenue (\$)
SC320	10.5	20.4	3.2	5.0	9.40
SC325	9.3	24.6	2.5	5.0	10.80
SC340	11.6	17.7	3.6	5.0	8.75
SC370	8.2	26.5	5.5	5.0	7.80
Daily capacity (minutes)	4800.0	9600.0	4700.0	4500.0	

The contractor guarantees a minimum production level of 100 units for each of the four cables.

   (a) Formulate the problem as a linear programming model, and determine the optimum production schedule.
   (b) Based on the dual prices, do you recommend making increases in the daily capacities of any of the four operations? Explain.
   (c) Does the minimum production requirements for the four cables represent an advantage or a disadvantage for NWAC Electronics? Provide an explanation based on the dual prices.
   (d) Can the present unit contribution to revenue as specified by the dual price be guaranteed if we increase the capacity of soldering by 10%?

3. BagCo produces leather jackets and handbags. A jacket requires 8 m^2 of leather, and a handbag only 2 m^2. The labor requirements for the two products are 12 and 5 hours, respectively. The current weekly supplies of leather and labor are limited to 1200 m^2 and 1850 hours. The company sells the jackets and handbags at \$350 and \$120, respectively. The objective is to determine the production schedule that maximizes the net revenue. BagCo is considering an expansion of production. What is the maximum purchase price the company should pay for additional leather? For additional labor?

4.3.2 Economic Interpretation of Dual Constraints

The dual constraints can be interpreted by using Formula 2 in Section 4.2.4, which states that at any primal iteration,

$$\text{Objective coefficient of } x_j = \begin{pmatrix}\text{Left-hand side of}\\ \text{dual constraint } j\end{pmatrix} - \begin{pmatrix}\text{Right-hand side of}\\ \text{dual constraint } j\end{pmatrix}$$

$$= \sum_{i=1}^{m} a_{ij}y_i - c_j$$

We use dimensional analysis once again to interpret this equation. The revenue per unit, c_j, of activity j is in dollars per unit. Hence, for consistency, the quantity $\sum_{i=1}^{m} a_{ij}y_i$ must also be in dollars per unit. Next, because c_j represents revenue, the quantity $\sum_{i=1}^{m} a_{ij}y_i$, which appears in the equation with an opposite sign, must represent cost. Thus we have

$$\$ \text{ cost} = \sum_{i=1}^{m} a_{ij}y_i = \sum_{i=1}^{m} \begin{pmatrix}\text{usage of resource } i\\ \text{per unit of activity } j\end{pmatrix} \times \begin{pmatrix}\text{cost per unit}\\ \text{of resource } i\end{pmatrix}$$

The conclusion here is that the dual variable y_i represents the **imputed cost** per unit of resource i, and we can think of the quantity $\sum_{i=1}^{m} a_{ij}y_i$ as the imputed cost of all the resources needed to produce one unit of activity j.

In Section 3.6, we referred to the quantity $\left(\sum_{i=1}^{m} a_{ij}y_i - c_j\right)$ as the **reduced cost** of activity j. The maximization optimality condition of the simplex method says that an increase in the level of an unused (nonbasic) activity j can improve revenue only if its reduced cost is negative. In terms of the preceding interpretation, this condition states that

$$\begin{pmatrix}\text{Imputed cost of}\\ \text{resources used by}\\ \text{one unit of activity } j\end{pmatrix} < \begin{pmatrix}\text{Revenue per unit}\\ \text{of activity } j\end{pmatrix}$$

The maximization optimality condition thus says that it is economically advantageous to increase an activity to a positive level if its unit revenue exceeds its unit imputed cost.

We will use the TOYCO model of Section 3.6 to demonstrate the computation. The details of the model are restated here for convenience.

Example 4.3-2

TOYCO assembles three types of toys: trains, trucks, and cars using three operations. Available assembly times for the three operations are 430, 460, and 420 minutes per day, respectively, and the revenues per toy train, truck, and car are \$3, \$2, and \$5, respectively. The assembly times per train for the three operations are 1, 3, and 1 minutes, respectively. The corresponding times per truck and per car are (2, 0, 4) and (1, 2, 0) minutes (a zero time indicates that the operation is not used).

Letting x_1, x_2, and x_3 represent the daily number of units assembled of trains, trucks and cars, the associated LP model and its dual are given as:

TOYCO primal	TOYCO dual
Maximize $z = 3x_1 + 2x_2 + 5x_3$	Minimize $w = 430y_1 + 460y_2 + 420y_3$
subject to	subject to
$x_1 + 2x_2 + x_3 \leq 430$ (Operation 1)	$y_1 + 3y_2 + y_3 \geq 3$
$3x_1 + 2x_3 \leq 460$ (Operation 2)	$2y_1 + 4y_3 \geq 2$
$x_1 + 4x_2 \leq 420$ (Operation 3)	$y_1 + 2y_2 \geq 5$
$x_1, x_2, x_3 \geq 0$	$y_1, y_2, y_3 \geq 0$
Optimal solution:	Optimal solution:
$x_1 = 0, x_2 = 100, x_3 = 230, z = \$1350$	$y_1 = 1, y_2 = 2, y_3 = 0, w = \$1350$

The optimal primal solution calls for producing no toy trains, 100 toy trucks, and 230 toy cars. Suppose that TOYCO is interested in producing toy trains as well. How can this be achieved? Looking at the problem from the standpoint of the interpretation of the *reduced cost* for x_1, toy trains will become attractive economically only if the imputed cost of the resources used to produce one toy train is strictly less than its unit revenue. TOYCO thus can either increase the unit revenue per unit by raising the unit price, or it can decrease the imputed cost of the used resources $(= y_1 + 3y_2 + y_3)$. An increase in unit price may not be possible because of market competition. A decrease in the unit imputed cost is more plausible because it entails making improvements in the assembly operations. Letting r_1, r_2, and r_3 represent the proportions by which the unit times of the three operations are reduced, the problem requires determining r_1, r_2, and r_3 such that the new imputed cost per per toy train is less than its unit revenue—that is,

$$1(1 - r_1)y_1 + 3(1 - r_2)y_2 + 1(1 - r_3)y_3 < 3$$

For the given optimal values of $y_1 = 1$, $y_2 = 2$, and $y_3 = 0$, this inequality reduces to (verify!)

$$r_1 + 6r_2 > 4$$

Thus, any values of r_1 and r_2 between 0 and 1 that satisfy $r_1 + 6r_2 > 4$ should make toy trains profitable. However, this goal may not be achievable because it requires practically impossible reductions in the times of operations 1 and 2. For example, even reductions as high as 50% in these times (that is, $r_1 = r_2 = .5$) fail to satisfy the given condition. Thus, TOYCO should not produce toy trains unless an increase in its unit price is possible.

PROBLEM SET 4.3B

1. In Example 4.3-2, suppose that for toy trains the per-unit time of operation 2 can be reduced from 3 minutes to at most 1.25 minutes. By how much must the per-unit time of operation 1 be reduced to make toy trains just profitable?

*2. In Example 4.3-2, suppose that TOYCO is studying the possibility of introducing a fourth toy: fire trucks. The assembly does not make use of operation 1. Its unit assembly times on operations 2 and 3 are 1 and 3 minutes, respectively. The revenue per unit is $4. Would you advise TOYCO to introduce the new product?

*3. JoShop uses lathes and drill presses to produce four types of machine parts, $PP1$, $PP2$, $PP3$, and $PP4$. The table below summarizes the pertinent data.

	Machining time in minutes per unit of				
Machine	*PP1*	*PP2*	*PP3*	*PP4*	Capacity (minutes)
Lathes	2	5	3	4	5300
Drill presses	3	4	6	4	5300
Unit revenue ($)	3	6	5	4	

For the parts that are not produced by the present optimum solution, determine the rate of deterioration in the optimum revenue per unit increase of each of these products.

4. Consider the optimal solution of JoShop in Problem 3. The company estimates that for each part that is not produced (per the optimum solution), an across-the-board 20% reduction in machining time can be realized through process improvements. Would these improvements make these parts profitable? If not, what is the minimum percentage reduction needed to realize revenueability?

4.4 ADDITIONAL SIMPLEX ALGORITHMS

In the simplex algorithm presented in Chapter 3 the problem starts at a (basic) feasible solution. Successive iterations continue to be feasible until the optimal is reached at the last iteration. The algorithm is sometimes referred to as the **primal simplex** method.

This section presents two additional algorithms: The **dual simplex** and the **generalized simplex**. In the dual simplex, the LP starts at a better than optimal *infeasible* (basic) solution. Successive iterations remain infeasible and (better than) optimal until feasibility is restored at the last iteration. The generalized simplex combines both the primal and dual simplex methods in one algorithm. It deals with problems that start both nonoptimal and infeasible. In this algorithm, successive iterations are associated with basic feasible or infeasible (basic) solutions. At the final iteration, the solution becomes optimal and feasible (assuming that one exists).

All three algorithms, the primal, the dual, and the generalized, are used in the course of post-optimal analysis calculations, as will be shown in Section 4.5.

4.4.1 Dual Simplex Algorithm

The crux of the dual simplex method is to start with a better than optimal and infeasible basic solution. The optimality and feasibility conditions are designed to preserve the optimality of the basic solutions while moving the solution iterations toward feasibility.

Dual feasibility condition. The leaving variable, x_r, is the basic variable having the most negative value (ties are broken arbitrarily). If all the basic variables are nonnegative, the algorithm ends.

Dual optimality condition. Given that x_r is the leaving variable, let $\bar{c}_j$ be the reduced cost of nonbasic variable x_j and α_{rj} the constraint coefficient in the x_r-row and x_j-column of the tableau. The entering variable is the nonbasic variable with $\alpha_{rj} < 0$ that corresponds to

$$\min_{\text{Nonbasic } x_j} \left\{ \left| \frac{\bar{c}_j}{\alpha_{rj}} \right|, \alpha_{rj} < 0 \right\}$$

(Ties are broken arbitrarily.) If $\alpha_{rj} \geq 0$ for all nonbasic x_j, the problem has no feasible solution.

To start the LP optimal and infeasible, two requirements must be met:

1. The objective function must satisfy the optimality condition of the regular simplex method (Chapter 3).
2. All the constraints must be of the type ($\leq$).

The second condition requires converting any ($\geq$) to ($\leq$) simply by multiplying both sides of the inequality ($\geq$) by -1. If the LP includes ($=$) constraints, the equation can be replaced by two inequalities. For example,

$$x_1 + x_2 = 1$$

is equivalent to

$$x_1 + x_2 \leq 1, x_1 + x_2 \geq 1$$

or

$$x_1 + x_2 \leq 1, -x_1 - x_2 \leq -1$$

After converting all the constraints to ($\leq$), the starting solution is infeasible if at least one of the right-hand sides of the inequalities is strictly negative.

Example 4.4-1

$$\text{Minimize } z = 3x_1 + 2x_2 + x_3$$

subject to

$$3x_1 + x_2 + x_3 \geq 3$$
$$-3x_1 + 3x_2 + x_3 \geq 6$$
$$x_1 + x_2 + x_3 \leq 3$$
$$x_1, x_2, x_3 \geq 0$$

In the present example, the first two inequalities are multiplied by -1 to convert them to ($\leq$) constraints. The starting tableau is thus given as:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-3	-2	-1	0	0	0	0
x_4	-3	-1	-1	1	0	0	-3
x_5	3	-3	-1	0	1	0	-6
x_6	1	1	1	0	0	1	3

The tableau is optimal because all the reduced costs in the z-row are ≤ 0 ($\bar{c}_1 = -3, \bar{c}_2 = -2, \bar{c}_3 = -1, \bar{c}_4 = 0, \bar{c}_5 = 0, \bar{c}_6 = 0$). It is also infeasible because at least one of the basic variables is negative ($x_4 = -3, x_5 = -6, x_6 = 3$).

According to the dual feasibility condition, x_5 ($= -6$) is the leaving variable. The next table shows how the dual optimality condition is used to determine the entering variable.

	$j = 1$	$j = 2$	$j = 3$
Nonbasic variable	x_1	x_2	x_3
z-row ($\bar{c}_j$)	-3	-2	-1
x_5-row, α_{4j}	3	-3	-1
Ratio, $\left\lvert\frac{\bar{c}_j}{\alpha_{5j}}\right\rvert$, $\alpha_{5j} < 0$	—	$\frac{2}{3}$	1

The ratios show that x_2 is the entering variable. Notice that a nonbasic variable x_j is a candidate for entering the basic solution only if its α_{rj} is strictly negative. This is the reason x_1 is excluded in the table above.

The next tableau is obtained by using the familiar row operations, which give

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-5	0	$-\frac{1}{3}$	0	$-\frac{2}{3}$	0	4
x_4	-4	0	$-\frac{2}{3}$	1	$-\frac{1}{3}$	0	-1
x_2	-1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	2
x_6	2	0	$\frac{2}{3}$	0	$\frac{1}{3}$	1	1
Ratio	$\frac{5}{4}$	—	$\frac{1}{2}$	—	2	—	

The preceding tableau shows that x_4 leaves and x_3 enters, thus yielding the following tableau, which is both optimal and feasible:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-3	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{9}{2}$
x_3	6	0	1	$-\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$
x_2	-3	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{3}{2}$
x_6	-2	0	0	1	0	1	0

Notice how the dual simplex works. In all the iterations, optimality is maintained (all reduced costs are ≤ 0). At the same time, each new iteration moves the solution toward feasibility. At iteration 3, feasibility is restored for the first time and the process ends with the optimal feasible solution given as $x_1 = 0, x_2 = \frac{3}{2}, x_2 = \frac{3}{2}$, and $z = \frac{9}{2}$.

TORA Moment.

TORA provides a tutorial module for the dual simplex method. From the SOLVE/MODIFY menu select Solve ⇒ Algebraic ⇒ Iterations ⇒ Dual Simplex. Remember that you need to convert (=) constraints to inequalities. You do not need to convert ($\geq$) constraints because TORA will do the conversion internally. If the LP does not satisfy the initial requirements of the dual simplex, a message will appear on the screen.

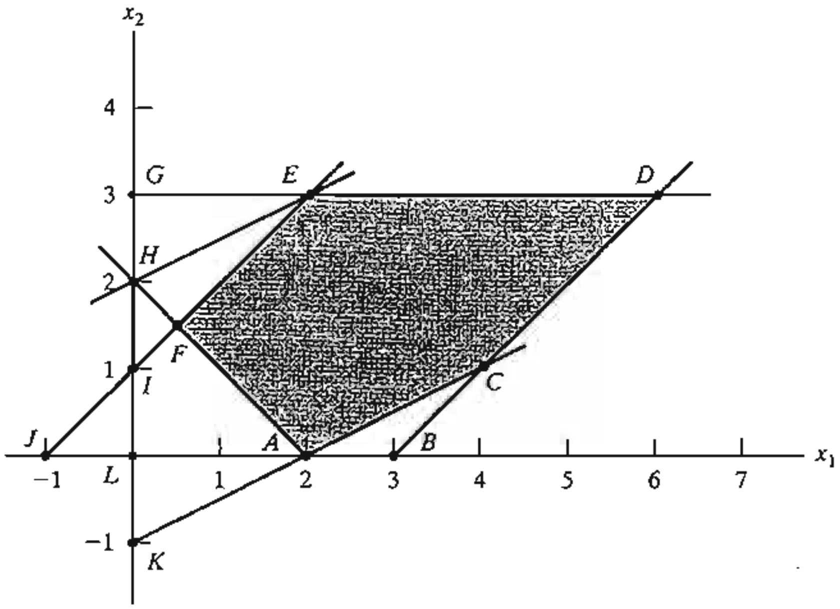
As in the regular simplex method, the tutorial module allows you to select the entering and the leaving variables beforehand. An appropriate feedback then tells you if your selection is correct.

PROBLEM SET 4.4A[2]

1. Consider the solution space in Figure 4.3, where it is desired to find the optimum extreme point that uses the *dual* simplex method to minimize $z = 2x_1 + x_2$. The optimal solution occurs at point $F = (0.5, 1.5)$ on the graph.
   (a) Can the dual simplex start at point A?
   *(b) If the starting basic (infeasible but better than optimum) solution is given by point G, would it be possible for the iterations of the dual simplex method to follow the path $G \rightarrow E \rightarrow F$? Explain.
   (c) If the starting basic (infeasible) solution starts at point L, identify a possible path of the dual simplex method that leads to the optimum feasible point at point F.
2. Generate the dual simplex iterations for the following problems (using TORA for convenience), and trace the path of the algorithm on the graphical solution space.
   (a) Minimize $z = 2x_1 + 3x_2$

FIGURE 4.3

Solution space for Problem 1, Set 4.4a



[2]You are encouraged to use TORA's tutorial mode where possible to avoid the tedious task of carrying out the Gauss-Jordan row operations. In this manner, you can concentrate on understanding the main ideas of the method.

subject to

$$2x_1 + 2x_2 \leq 30$$
$$x_1 + 2x_2 \geq 10$$
$$x_1, x_2 \geq 0$$

(b) Minimize $z = 5x_1 + 6x_2$
subject to

$$x_1 + x_2 \geq 2$$
$$4x_1 + x_2 \geq 4$$
$$x_1, x_2 \geq 0$$

(c) Minimize $z = 4x_1 + 2x_2$
subject to

$$x_1 + x_2 = 1$$
$$3x_1 - x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

(d) Minimize $z = 2x_1 + 3x_2$
subject to

$$2x_1 + x_2 \geq 3$$
$$x_1 + x_2 = 2$$
$$x_1, x_2 \geq 0$$

3. *Dual Simplex with Artificial Constraints.* Consider the following problem:

$$\text{Maximize } z = 2x_1 - x_2 + x_3$$

subject to

$$2x_1 + 3x_2 - 5x_3 \geq 4$$
$$-x_1 + 9x_2 - x_3 \geq 3$$
$$4x_1 + 6x_2 + 3x_3 \leq 8$$
$$x_1, x_2, x_3 \geq 0$$

The starting basic solution consisting of surplus variables x_4 and x_5 and slack variable x_6 is infeasible because $x_4 = -4$ and $x_5 = -3$. However, the dual simplex is not applicable directly, because x_1 and x_3 do not satisfy the maximization optimality condition. Show that by adding the artificial constraint $x_1 + x_3 \leq M$ (where M is sufficiently large not to eliminate any feasible points in the original solution space), and then using the new constraint as a pivot row, the selection of x_1 as the entering variable (because it has the most negative objective coefficient) will render an all-optimal objective row. Next, carry out the regular dual simplex method on the modified problem.

4. Using the artificial constraint procedure introduced in Problem 3, solve the following problems by the dual simplex method. In each case, indicate whether the resulting solution is feasible, infeasible, or unbounded.

(a) Maximize $z = 2x_3$
subject to

$$\begin{aligned} -x_1 + 2x_2 - 2x_3 &\geq 8 \\ -x_1 + x_2 + x_3 &\leq 4 \\ 2x_1 - x_2 + 4x_3 &\leq 10 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(b) Maximize $z = x_1 - 3x_2$
subject to

$$\begin{aligned} x_1 - x_2 &\leq 2 \\ x_1 + x_2 &\geq 4 \\ 2x_1 - 2x_2 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

***(c)** Minimize $z = -x_1 + x_2$
subject to

$$\begin{aligned} x_1 - 4x_2 &\geq 5 \\ x_1 - 3x_2 &\leq 1 \\ 2x_1 - 5x_2 &\geq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(d) Maximize $z = 2x_3$
subject to

$$\begin{aligned} -x_1 + 3x_2 - 7x_3 &\geq 5 \\ -x_1 + x_2 - x_3 &\leq 1 \\ 3x_1 + x_2 - 10x_3 &\leq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

5. Solve the following LP in three different ways (use TORA for convenience). Which method appears to be the most efficient computationally?

$$\text{Minimize } z = 6x_1 + 7x_2 + 3x_3 + 5x_4$$

subject to

$$\begin{aligned} 5x_1 + 6x_2 - 3x_3 + 4x_4 &\geq 12 \\ x_2 - 5x_3 - 6x_4 &\geq 10 \\ 2x_1 + 5x_2 + x_3 + x_4 &\geq 8 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

4.4.2 Generalized Simplex Algorithm

The (primal) simplex algorithm in Chapter 3 starts feasible but nonoptimal. The dual simplex in Section 4.4.1 starts (better than) optimal but infeasible. What if an LP model starts both nonoptimal and infeasible? We have seen that the primal simplex accounts for the infeasibility of the starting solution by using artificial variables. Similarly, the dual simplex accounts for the nonoptimality by using an artificial constraint (see Problem 3, Set 4.4a). Although these procedures are designed to enhance *automatic* computations, such details may cause one to lose sight of what the simplex algorithm truly entails—namely, the optimum solution of an LP is associated with a corner point (or basic) solution. Based on this observation, you should be able to "tailor" your own simplex algorithm for LP models that start both nonoptimal and infeasible. The following example illustrates what we call the generalized simplex algorithm.

Example 4.4-2

Consider the LP model of Problem 4(a), Set 4.4a. The model can be put in the following tableau form in which the starting basic solution (x_3, x_4, x_5) is both nonoptimal (because x_3 has a negative reduced cost) and infeasible (because $x_4 = -8$). (The first equation has been multiplied by -1 to reveal the infeasibility directly in the *Solution* column.) 4.5

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	-2	0	0	0	0
x_4	1	-2	2	1	0	0	-8
x_5	-1	1	1	0	1	0	4
x_6	2	-1	4	0	0	1	10

We can solve the problem without the use of any artificial variables or artificial constraints as follows: Remove infeasibility first by applying a version of the dual simplex feasibility condition that selects x_4 as the leaving variable. To determine the entering variable, all we need is a nonbasic variable whose constraint coefficient in the x_4-row is strictly negative. The selection can be done without regard to optimality, because it is nonexistent at this point anyway (compare with the dual optimality condition). In the present example, x_2 has a negative coefficient in the x_4-row and is selected as the entering variable. The result is the following tableau:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	-2	0	0	0	0
x_2	$-\frac{1}{2}$	1	-1	$-\frac{1}{2}$	0	0	4
x_5	$-\frac{1}{2}$	0	2	$\frac{1}{2}$	1	0	0
x_6	$\frac{3}{2}$	0	3	$-\frac{1}{2}$	0	1	14

The solution in the preceding tableau is now feasible but nonoptimal, and we can use the primal simplex to determine the optimal solution. In general, had we not restored feasibility in the preceding tableau, we would repeat the procedure as necessary until feasibility is satisfied or there is evidence that the problem has no feasible solution (which happens if a basic variable is negative and all its constraint coefficients are nonnegative). Once feasibility is established, the next step is to pay attention to optimality by applying the proper optimality condition of the primal simplex method.

Remarks. The essence of Example 4.4-2 is that the simplex method is not rigid. The literature abounds with variations of the simplex method (e.g., the primal-dual method, the symmetrical method, the criss-cross method, and the multiplex method) that give the impression that each procedure is different, when, in effect, they all seek a corner point solution, with a slant toward automated computations and, perhaps, computational efficiency.

PROBLEM SET 4.4B

1. The LP model of Problem 4(c), Set 4.4a, has no feasible solution. Show how this condition is detected by the *generalized simplex procedure*.
2. The LP model of Problem 4(d), Set 4.4a, has no bounded solution. Show how this condition is detected by the *generalized simplex procedure*.

4.5 POST-OPTIMAL ANALYSIS

In Section 3.6, we dealt with the sensitivity of the optimum solution by determining the ranges for the different parameters that would keep the optimum basic solution unchanged. In this section, we deal with making changes in the parameters of the model and finding the new optimum solution. Take, for example, a case in the poultry industry where an LP model is commonly used to determine the optimal feed mix per broiler (see Example 2.2-2). The weekly consumption per broiler varies from .26 lb (120 grams) for a one-week-old bird to 2.1 lb (950 grams) for an eight-week-old bird. Additionally, the cost of the ingredients in the mix may change periodically. These changes require periodic recalculation of the optimum solution. *Post-optimal analysis* determines the new solution in an efficient way. The new computations are rooted in the use duality and the primal-dual relationships given in Section 4.2.

The following table lists the cases that can arise in post-optimal analysis and the actions needed to obtain the new solution (assuming one exists):

Condition after parameters change	Recommended action
Current solution remains optimal and feasible.	No further action is necessary.
Current solution becomes infeasible.	Use dual simplex to recover feasibility.
Current solution becomes nonoptimal.	Use primal simplex to recover optimality.
Current solution becomes both nonoptimal and infeasible.	Use the generalized simplex method to obtain new solution.

The first three cases are investigated in this section. The fourth case, being a combination of cases 2 and 3, is treated in Problem 6, Set 4.5a.

The TOYCO model of Example 4.3-2 will be used to explain the different procedures. Recall that the TOYCO model deals with the assembly of three types of toys: trains, trucks, and cars. Three operations are involved in the assembly. We wish to determine the number of units of each toy that will maximize revenue. The model and its dual are repeated here for convenience.

TOYCO primal	TOYCO dual
Maximize $z = 3x_1 + 2x_2 + 5x_3$	Minimize $z = 430y_1 + 460y_2 + 420y_3$
subject to	subject to
$x_1 + 2x_2 + x_3 \le 430$ (Operation 1)	$y_1 + 3y_2 + y_3 \ge 3$
$3x_1 + 2x_3 \le 460$ (Operation 2)	$2y_1 + 4y_3 \ge 2$
$x_1 + 4x_2 \le 420$ (Operation 3)	$y_1 + 2y_2 \ge 5$
$x_1, x_2, x_3 \ge 0$	$y_1, y_2, y_3 \ge 0$
Optimal solution:	Optimal solution:
$x_1 = 0, x_2 = 100, x_3 = 230, z = \$1350$	$y_1 = 1, y_2 = 2, y_3 = 0, w = \$1350$

The associated optimum tableau for the primal is given as

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

4.5.1 Changes Affecting Feasibility

The feasibility of the current optimum solution may be affected only if (1) the right-hand side of the constraints is changed, or (2) a new constraint is added to the model. In both cases, infeasibility occurs when at least one element of the right-hand side of the optimal tableau becomes negative—that is, one or more of the current basic variables become negative.

Changes in the right-hand side. This change requires recomputing the right-hand side of the tableau using Formula 1 in Section 4.2.4:

$$\begin{pmatrix}\text{New right-hand side of}\\ \text{tableau in iteration } i\end{pmatrix} = \begin{pmatrix}\text{Inverse in}\\ \text{iteration } i\end{pmatrix} \times \begin{pmatrix}\text{New right-hand}\\ \text{side of constraints}\end{pmatrix}$$

Recall that the right-hand side of the tableau gives the values of the basic variables.

Example 4.5-1

Situation 1. Suppose that TOYCO wants to expand its assembly lines by increasing the daily capacity of operations 1, 2, and 3 by 40% to 602, 644, and 588 minutes, respectively. How would this change affect the total revenue?

With these increases, the only change that will take place in the optimum tableau is the right-hand side of the constraints (and the optimum objective value). Thus, the new basic solution is computed as follows:

$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 602 \\ 644 \\ 588 \end{pmatrix} = \begin{pmatrix} 140 \\ 322 \\ 28 \end{pmatrix}$$

Thus, the current basic variables, x_2, x_3, and x_6, remain feasible at the new values 140, 322, and 28, respectively. The associated optimum revenue is \$1890, which is \$540 more than the current revenue of \$1350.

Situation 2. Although the new solution is appealing from the standpoint of increased revenue, TOYCO recognizes that its implementation may take time. Another proposal was thus made to shift the slack capacity of operation 3 ($x_6 = 20$ minutes) to the capacity of operation 1. How would this change impact the optimum solution?

The capacity mix of the three operations changes to 450, 460, and 400 minutes, respectively. The resulting solution is

$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 450 \\ 460 \\ 400 \end{pmatrix} = \begin{pmatrix} 110 \\ 230 \\ -40 \end{pmatrix}$$

The resulting solution is infeasible because $x_6 = -40$, which requires applying the dual simplex method to recover feasibility. First, we modify the right-hand side of the tableau as shown by the shaded column. Notice that the associated value of $z = 3 \times 0 + 2 \times 110 + 5 \times 230 = \$1370$.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1370
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	110
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	-40

From the dual simplex, x_6 leaves and x_4 enters, which yields the following optimal feasible tableau (in general, the dual simplex may take more than one iteration to recover feasibility).

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	5	0	0	0	$\frac{5}{2}$	$\frac{1}{2}$	1350
x_2	$\frac{1}{4}$	1	0	0	0	$\frac{1}{4}$	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_4	-1	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	20

The optimum solution (in terms of x_1, x_2, and x_3) remains the same as in the original model. This means that the proposed shift in capacity allocation is not advantageous in this case because all it does is shift the surplus capacity in operation 3 to a surplus capacity in operation 1. The conclusion is that operation 2 is the bottleneck and it may be advantageous to shift the surplus to operation 2 instead (see Problem 1, Set 4.5a). The selection of operation 2 over operation 1 is also reinforced by the fact that the dual price for operation 2 ($2/min) is higher than that for operation 2 (= $1/min).

PROBLEM SET 4.5A

1. In the TOYCO model listed at the start of Section 4.5, would it be more advantageous to assign the 20-minute excess capacity of operation 3 to operation 2 instead of operation 1?

2. Suppose that TOYCO wants to change the capacities of the three operations according to the following cases:

$$\textbf{(a)} \begin{pmatrix}460\\500\\400\end{pmatrix} \quad \textbf{(b)} \begin{pmatrix}500\\400\\600\end{pmatrix} \quad \textbf{(c)} \begin{pmatrix}300\\800\\200\end{pmatrix} \quad \textbf{(d)} \begin{pmatrix}450\\700\\350\end{pmatrix}$$

Use post-optimal analysis to determine the optimum solution in each case.

3. Consider the Reddy Mikks model of Example 2.1-1. Its optimal tableau is given in Example 3.3-1. If the daily availabilities of raw materials $M1$ and $M2$ are increased to 28 and 8 tons, respectively, use post-optimal analysis to determine the new optimal solution.

***4.** The Ozark Farm has 20,000 broilers that are fed for 8 weeks before being marketed. The weekly feed per broiler varies according to the following schedule:

Week	1	2	3	4	5	6	7	8
lb/broiler	.26	.48	.75	1.00	1.30	1.60	1.90	2.10

For the broiler to reach a desired weight gain in 8 weeks, the feedstuffs must satisfy specific nutritional needs. Although a typical list of feedstuffs is large, for simplicity we will limit the model to three items only: limestone, corn, and soybean meal. The nutritional needs will also be limited to three types: calcium, protein, and fiber. The following table summarizes the nutritive content of the selected ingredients together with the cost data.

	Content (lb) per lb of			
Ingredient	*Calcium*	*Protein*	*Fiber*	$ per lb
Limestone	.380	.00	.00	.12
Corn	.001	.09	.02	.45
Soybean meal	.002	.50	.08	1.60

The feed mix must contain

(a) At least .8% but not more than 1.2% calcium

(b) At least 22% protein

(c) At most 5% crude fiber

Solve the LP for week 1 and then use post-optimal analysis to develop an optimal schedule for the remaining 7 weeks.

5. Show that the 100% feasibility rule in Problem 12, Set 3.6c (Chapter 3) is based on the condition

$$\begin{pmatrix}\text{Optimum}\\ \text{inverse}\end{pmatrix}\begin{pmatrix}\text{Original right-hand}\\ \text{side vector}\end{pmatrix} \geq 0$$

6. *Post-optimal Analysis for Cases Affecting Both Optimality and Feasibility.* Suppose that you are given the following simultaneous changes in the Reddy Mikks model: The revenue per ton of exterior and interior paints are \$1000 and \$4000, respectively, and the maximum daily availabilities of raw materials, $M1$ and $M2$, are 28 and 8 tons, respectively.
   (a) Show that the proposed changes will render the current optimal solution both nonoptimal and infeasible.
   (b) Use the *generalized simplex algorithm* (Section 4.4.2) to determine the new optimal feasible solution.

Addition of New Constraints. The addition of a new constraint to an existing model can lead to one of two cases.

1. The new constraint is *redundant*, meaning that it is satisfied by the current optimum solution, and hence can be dropped from the model altogether.
2. The current solution violates the new constraint, in which case the dual simplex method is used to restore feasibility.

Notice that the addition of a new constraint can never improve the current optimum objective value.

Example 4.5-3

Situation 1. Suppose that TOYCO is changing the design of its toys, and that the change will require the addition of a fourth operation in the assembly lines. The daily capacity of the new operation is 500 minutes and the times per unit for the three products on this operation are 3, 1, and 1 minutes, respectively. Study the effect of the new operation on the optimum solution.

The constraint for operation 4 is

$$3x_1 + x_2 + x_3 \leq 500$$

This constraint is redundant because it is satisfied by the current optimum solution $x_1 = 0$, $x_2 = 100$, and $x_3 = 230$. Hence, the current optimum solution remains unchanged.

Situation 2. Suppose, instead, that TOYCO unit times on the fourth operation are changed to 3, 3, and 1 minutes, respectively. All the remaining data of the model remain the same. Will the optimum solution change?

The constraint for operation 4 is

$$3x_1 + 3x_2 + x_3 \leq 500$$

This constraint is not satisfied by the current optimum solution. Thus, the new constraint must be added to the current optimum tableau as follows (x_7 is a slack):

Basic	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Solution
z	4	0	0	1	2	0	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	0	230
x_6	2	0	0	-2	1	1	0	20
x_7	3	3	1	0	0	0	1	500

The tableau shows that $x_7 = 500$, which is not consistent with the values of x_2 and x_3 in the rest of the tableau. The reason is that the basic variables x_2 and x_3 have not been substituted out in the new constraint. This substitution is achieved by performing the following operation:

$$\text{New } x_7\text{-row} = \text{Old } x_7\text{-row} - \{3 \times (x_2\text{-row}) + 1 \times (x_3\text{-row})\}$$

This operation is exactly the same as substituting

$$x_2 = 100 - \left(-\tfrac{1}{4}x_1 + \tfrac{1}{2}x_4 - \tfrac{1}{4}x_5\right)$$
$$x_3 = 230 - \left(\tfrac{3}{2}x_1 + \tfrac{1}{2}x_5\right)$$

in the new constraint. The new tableau is thus given as

Basic	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Solution
z	4	0	0	1	2	0	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	0	230
x_6	2	0	0	-2	1	1	0	20
x_7	$\frac{9}{4}$	0	0	$-\frac{3}{2}$	$\frac{1}{4}$	0	1	-30

Application of the dual simplex method will produce the new optimum solution $x_1 = 0$, $x_2 = 90$, $x_3 = 230$, and $z = \$1330$ (verify!). The solution shows that the addition of operation 4 will worsen the revenues from \$1350 to \$1330.

PROBLEM SET 4.5B

1. In the TOYCO model, suppose the fourth operation has the following specifications: The maximum production rate based on 480 minutes a day is either 120 units of product 1, 480 units of product 2, or 240 units of product 3. Determine the optimal solution, assuming that the daily capacity is limited to
   - ***(a)** 570 minutes.
   - **(b)** 548 minutes.
2. *Secondary Constraints.* Instead of solving a problem using all of its constraints, we can start by identifying the so-called *secondary constraints.* These are the constraints that we

suspect are least restrictive in terms of the optimum solution. The model is solved using the remaining (primary) constraints. We may then add the secondary constraints one at a time. A secondary constraint is discarded if it satisfies the available optimum. The process is repeated until all the secondary constraints are accounted for.

Apply the proposed procedure to the following LP:

$$\text{Maximize } z = 5x_1 + 6x_2 + 3x_2$$

subject to

$$\begin{aligned} 5x_1 + 5x_2 + 3x_3 &\le 50 \\ x_1 + x_2 - x_3 &\le 20 \\ 7x_1 + 6x_2 - 9x_3 &\le 30 \\ 5x_1 + 5x_2 + 5x_3 &\le 35 \\ 12x_1 + 6x_2 \quad\quad &\le 90 \\ x_2 - 9x_3 &\le 20 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

4.5.2 Changes Affecting Optimality

This section considers two particular situations that could affect the optimality of the current solution:

1. Changes in the original objective coefficients.
2. Addition of a new economic activity (variable) to the model.

Changes in the Objective Function Coefficients. These changes affect only the optimality of the solution. Such changes thus require recomputing the z-row coefficients (reduced costs) according to the following procedure:

1. Compute the dual values using Method 2 in Section 4.2.3.
2. Use the new dual values in Formula 2, Section 4.2.4, to determine the new reduced costs (z-row coefficients).

Two cases will result:

1. New z-row satisfies the optimality condition. The solution remains unchanged (the optimum objective value may change, however).
2. The optimality condition is not satisfied. Apply the (primal) simplex method to recover optimality.

Example 4.5-4

Situation 1. In the TOYCO model, suppose that the company has a new pricing policy to meet the competition. The unit revenues under the new policy are \$2, \$3, and \$4 for train, truck, and car toys, respectively. How is the optimal solution affected?

The new objective function is

$$\text{Maximize } z = 2x_1 + 3x_2 + 4x_3$$

Thus,

$$(\text{New objective coefficients of basic } x_2, x_3, \text{ and } x_6) = (3, 4, 0)$$

Using Method 2, Section 4.2.3, the dual variables are computed as

$$(y_1, y_2, y_3) = (3, 4, 0)\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} = \left(\tfrac{3}{2}, \tfrac{5}{4}, 0\right)$$

The z-row coefficients are determined as the difference between the left- and right-hand sides of the dual constraints (Formula 2, Section 4.2.4). It is not necessary to recompute the objective-row coefficients of the basic variables x_2, x_3, and x_6 because they always equal zero regardless of any changes made in the objective coefficients (verify!).

$$\text{Reduced cost of } x_1 = y_1 + 3y_2 + y_3 - \mathbf{2} = \tfrac{3}{2} + 3\left(\tfrac{5}{4}\right) + 0 - \mathbf{2} = \tfrac{13}{4}$$

$$\text{Reduced cost of } x_4 = y_1 - 0 = \tfrac{3}{2}$$

$$\text{Reduced cost of } x_5 = y_2 - 0 = \tfrac{5}{4}$$

Note that the right-hand side of the first dual constraint is 2, the *new* coefficient in the modified objective function.

The computations show that the current solution, $x_1 = 0$ train, $x_2 = 100$ trucks, and $x_3 = 230$ cars, remains optimal. The corresponding new revenue is computed as $2 \times 0 + 3 \times 100 + 4 \times 230 = \$1220$. The new pricing policy is not advantageous because it leads to lower revenue.

Situation 2. Suppose now that the TOYCO objective function is changed to

$$\text{Maximize } z = 6x_1 + 3x_2 + 4x_3$$

Will the optimum solution change?

We have

$$(y_1, y_2, y_3) = (3, 4, 0)\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} = \left(\tfrac{3}{2}, \tfrac{5}{4}, 0\right)$$

$$\text{Reduced cost of } x_1 = y_1 + 3y_2 + y_3 - \mathbf{6} = \tfrac{3}{2} + 3\left(\tfrac{5}{4}\right) + 0 - \mathbf{6} = -\tfrac{3}{4}$$

$$\text{Reduced cost of } x_4 = y_1 - 0 = \tfrac{3}{2}$$

$$\text{Reduced cost of } x_5 = y_2 - 0 = \tfrac{5}{4}$$

The new reduced cost of x_1 shows that the current solution is not optimum.

To determine the new solution, the z-row is changed as highlighted in the following tableau:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$-\frac{3}{4}$	0	0	$\frac{3}{2}$	$\frac{5}{4}$	0	1220
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

The elements shown in the shaded cells are the new *reduced cost* for the nonbasic variables x_1, x_4, and x_5. All the remaining elements are the same as in the original optimal tableau. The new optimum solution is then determined by letting x_1 enter and x_6 leave, which yields $x_1 = 10, x_2 = 102.5, x_3 = 215$, and $z = \$1227.50$ (verify!). Although the new solution recommends the production of all three toys, the optimum revenue is less than when two toys only are manufactured.

PROBLEM SET 4.5C

1. Investigate the optimality of the TOYCO solution for each of the following objective functions. If the solution changes, use post-optimal analysis to determine the new optimum. (The optimum tableau of TOYCO is given at the start of Section 4.5.)
   (a) $z = 2x_1 + x_2 + 4x_3$
   (b) $z = 3x_1 + 6x_2 + x_3$
   (c) $z = 8x_1 + 3x_2 + 9x_3$
2. Investigate the optimality of the Reddy Mikks solution (Example 4.3-1) for each of the following objective functions. If the solution changes, use post-optimal analysis to determine the new optimum. (The optimal tableau of the model is given in Example 3.3-1.)
   ***(a)** $z = 3x_1 + 2x_2$
   (b) $z = 8x_1 + 10x_2$
   ***(c)** $z = 2x_1 + 5x_2$
3. Show that the 100% optimality rule (Problem 8, Set 3.6d, Chapter 3) is derived from (reduced costs) ≥ 0 for maximization problems and (reduced costs) ≤ 0 for minimization problems.

Addition of a New Activity. The addition of a new activity in an LP model is equivalent to adding a new variable. Intuitively, the addition of a new activity is desirable only if it is profitable—that is, if it improves the optimal value of the objective function. This condition can be checked by computing the reduced cost of the new variable using Formula 2, Section 4.2.4. If the new activity satisfies the optimality condition, then the activity is not profitable. Else, it is advantageous to undertake the new activity.

Example 4.5-5

TOYCO recognizes that toy trains are not currently in production because they are not profitable. The company wants to replace toy trains with a new product, a toy fire engine, to be assembled on the existing facilities. TOYCO estimates the revenue per toy fire engine to be \$4 and the assembly times per unit to be 1 minute on each of operations 1 and 2, and 2 minutes on operation 3. How would this change impact the solution?

Let x_7 represent the new fire engine product. Given that $(y_1, y_2, y_3) = (1, 2, 0)$ are the optimal dual values, we get

$$\text{Reduced cost of } x_7 = 1y_1 + 1y_2 + 2y_3 - 4 = 1 \times 1 + 1 \times 2 + 2 \times 0 - 4 = -1$$

The result shows that it is profitable to include x_7 in the optimal basic solution. To obtain the new optimum, we first compute its column constraint using Formula 1, Section 4.2.4, as

$$x_7\text{-constraint colum} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

Thus, the current simplex tableau must be modified as follows

Basic	x_1	x_2	x_3	x_7	x_4	x_5	x_6	Solution
z	4	0	0	-1	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	230
x_6	2	0	0	1	-2	1	1	20

The new optimum is determined by letting x_7 enter the basic solution, in which case x_6 must leave. The new solution is $x_1 = 0, x_2 = 0, x_3 = 125, x_7 = 210$, and $z = \$1465$ (verify!), which improves the revenues by \$115.

PROBLEM SET 4.5D

*1. In the original TOYCO model, toy trains are not part of the optimal product mix. The company recognizes that market competition will not allow raising the unit price of the toy. Instead, the company wants to concentrate on improving the assembly operation itself. This entails reducing the assembly time per unit in each of the three operations by a specified percentage, $p\%$. Determine the value of p that will make toy trains just profitable. (The optimum tableau of the TOYCO model is given at the start of Section 4.5.)

2. In the TOYCO model, suppose that the company can reduce the unit times on operations 1, 2, and 3 for toy trains from the current levels of 1, 3, and 1 minutes to .5, 1, and .5 minutes, respectively. The revenue per unit remains unchanged at \$3. Determine the new optimum solution.

3. In the TOYCO model, suppose that a new toy (fire engine) requires 3, 2, 4 minutes, respectively, on operations 1, 2, and 3. Determine the optimal solution when the revenue per unit is given by

   *(a) \$5.

   (b) \$10.

4. In the Reddy Mikks model, the company is considering the production of a cheaper brand of exterior paint whose input requirements per ton include .75 ton of each of raw materials $M1$ and $M2$. Market conditions still dictate that the excess of interior paint over the production of *both* types of exterior paint be limited to 1 ton daily. The revenue per ton of the new exterior paint is $3500. Determine the new optimal solution. (The model is explained in Example 4.5-1, and its optimum tableau is given in Example 3.3-1.)

REFERENCES

Bradley, S., A. Hax, and T. Magnanti, *Applied Mathematical Programming*, Addison-Wesley, Reading, MA, 1977.

Bazaraa, M., J. Jarvis, and H. Sherali, *Linear Programming and Network Flows*, 2nd ed., Wiley, New York, 1990.

Diwckar, U., *Introduction to Applied Optimization*, Kluwer Academic Publishers, Boston, 2003.

Nering, E., and A. Tucker, *Linear Programming and Related Problems*, Academic Press, Boston, 1992.

Vanderbei, R., *Linear Programming: Foundation and Optimization*, 2nd ed., Kluwer Academic Publishers, Boston MA, 2001.

CHAPTER 5

Transportation Model and Its Variants

Chapter Guide. The transportation model is a special class of linear programs that deals with shipping a commodity from *sources* (e.g., factories) to *destinations* (e.g., warehouses). The objective is to determine the shipping schedule that minimizes the total shipping cost while satisfying supply and demand limits. The application of the transportation model can be extended to other areas of operation, including inventory control, employment scheduling, and personnel assignment.

As you study the material in this chapter, keep in mind that the steps of the transportation algorithm are precisely those of the simplex method. Another point is that the transportation algorithm was developed in the early days of OR to enhance hand computations. Now, with the tremendous power of the computer, such shortcuts may not be warranted and, indeed, are never used in commercial codes in the strict manner presented in this chapter. Nevertheless, the presentation shows that the special transportation tableau is useful in modeling a class of problems in a concise manner (as opposed to the familiar LP model with explicit objective function and constraints). In particular, the transportation tableau format simplifies the solution of the problem by Excel Solver. The representation also provides interesting ideas about how the basic theory of linear programming is exploited to produce shortcuts in computations.

You will find TORA's tutorial module helpful in understanding the details of the transportation algorithm. The module allows you to make the decisions regarding the logic of the computations with immediate feedback.

This chapter includes a summary of 1 real-life application, 12 solved examples, 1 Solver model, 4 AMPL models, 46 end-of-section problems, and 5 cases. The cases are in Appendix E on the CD. The AMPL/Excel/Solver/TORA programs are in folder ch5Files.

Real-life Application—Scheduling Appointments at Australian Trade Events

The Australian Tourist Commission (ATC) organizes trade events around the world to provide a forum for Australian sellers to meet international buyers of tourism products, including accommodation, tours, and transport. During these events, sellers are

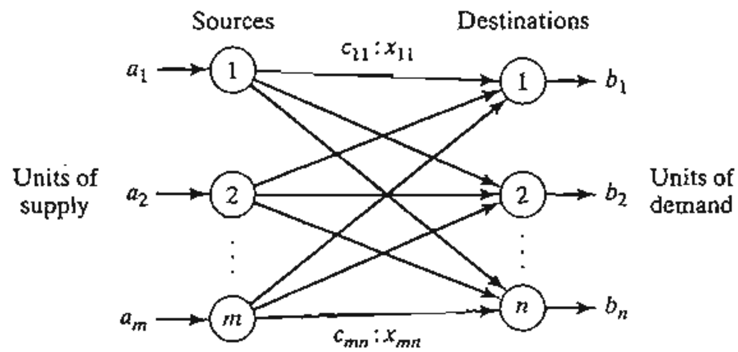


FIGURE 5.1
Representation of the transportation model with nodes and arcs

stationed in booths and are visited by buyers according to scheduled appointments. Because of the limited number of time slots available in each event and the fact that the number of buyers and sellers can be quite large (one such event held in Melbourne in 1997 attracted 620 sellers and 700 buyers), ATC attempts to schedule the seller-buyer appointments in advance of the event in a manner that maximizes preferences. The model has resulted in greater satisfaction for both the buyers and sellers. Case 3 in Chapter 24 on the CD provides the details of the study.

5.1 DEFINITION OF THE TRANSPORTATION MODEL

The general problem is represented by the network in Figure 5.1. There are m sources and n destinations, each represented by a **node**. The **arcs** represent the routes linking the sources and the destinations. Arc (i, j) joining source i to destination j carries two pieces of information: the transportation cost per unit, c_{ij}, and the amount shipped, x_{ij}. The amount of supply at source i is a_i and the amount of demand at destination j is b_j. The objective of the model is to determine the unknowns x_{ij} that will minimize the total transportation cost while satisfying all the supply and demand restrictions.

Example 5.1-1

MG Auto has three plants in Los Angeles, Detroit, and New Orleans, and two major distribution centers in Denver and Miami. The capacities of the three plants during the next quarter are 1000, 1500, and 1200 cars. The quarterly demands at the two distribution centers are 2300 and 1400 cars. The mileage chart between the plants and the distribution centers is given in Table 5.1.

The trucking company in charge of transporting the cars charges 8 cents per mile per car. The transportation costs per car on the different routes, rounded to the closest dollar, are given in Table 5.2.

The LP model of the problem is given as

$$\text{Minimize } z = 80x_{11} + 215x_{12} + 100x_{21} + 108x_{22} + 102x_{31} + 68x_{32}$$

TABLE 5.1 Mileage Chart

	Denver	Miami
Los Angeles	1000	2690
Detroit	1250	1350
New Orleans	1275	850

TABLE 5.2 Transportation Cost per Car

	Denver (1)	Miami (2)
Los Angeles (1)	$80	$215
Detroit (2)	$100	$108
New Orleans (3)	$102	$68

subject to

$$\begin{array}{llllllll}
x_{11} + x_{12} & & & = 1000 & \text{(Los Angeles)} \\
& x_{21} + x_{22} & & = 1500 & \text{(Detroit)} \\
& & + x_{31} + x_{32} & = 1200 & \text{(New Oreleans)} \\
x_{11} & + x_{21} + & x_{31} & = 2300 & \text{(Denver)} \\
x_{12} + & x_{22} & + x_{32} & = 1400 & \text{(Miami)} \\
\end{array}$$

$$x_{ij} \geq 0, i = 1, 2, 3, j = 1, 2$$

These constraints are all equations because the total supply from the three sources (= 1000 + 1500 + 1200 = 3700 cars) equals the total demand at the two destinations (= 2300 + 1400 = 3700 cars).

The LP model can be solved by the simplex method. However, with the special structure of the constraints we can solve the problem more conveniently using the **transportation tableau** shown in Table 5.3.

TABLE 5.3 MG Transportation Model

	Denver	Miami	Supply
Los Angeles	80 x_{11}	215 x_{12}	**1000**
Detroit	100 x_{21}	108 x_{22}	**1500**
New Orleans	102 x_{31}	68 x_{32}	**1200**
Demand	**2300**	**1400**	

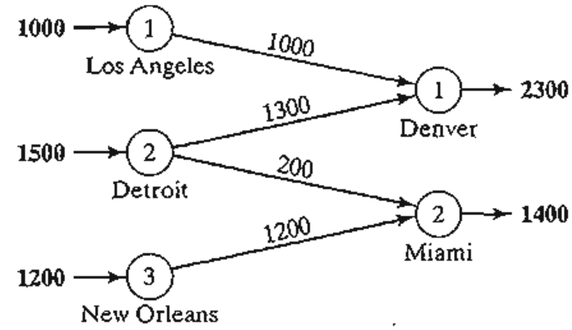


FIGURE 5.2
Optimal solution of MG Auto model

The optimal solution in Figure 5.2 (obtained by TORA[1]) calls for shipping 1000 cars from Los Angeles to Denver, 1300 from Detroit to Denver, 200 from Detroit to Miami, and 1200 from New Orleans to Miami. The associated minimum transportation cost is computed as $1000 \times \$80 + 1300 \times \$100 + 200 \times \$108 + 1200 \times \$68 = \$313{,}200$.

Balancing the Transportation Model. The transportation algorithm is based on the assumption that the model is balanced, meaning that the total demand equals the total supply. If the model is unbalanced, we can always add a dummy source or a dummy destination to restore balance.

Example 5.1-2

In the MG model, suppose that the Detroit plant capacity is 1300 cars (instead of 1500). The total supply (= 3500 cars) is less than the total demand (= 3700 cars), meaning that part of the demand at Denver and Miami will not be satisfied.

Because the demand exceeds the supply, a dummy source (plant) with a capacity of 200 cars (= 3700 − 3500) is added to balance the transportation model. The unit transportation costs from the dummy plant to the two destinations are zero because the plant does not exist.

Table 5.4 gives the balanced model together with its optimum solution. The solution shows that the dummy plant ships 200 cars to Miami, which means that Miami will be 200 cars short of satisfying its demand of 1400 cars.

We can make sure that a specific destination does not experience shortage by assigning a very high unit transportation cost from the dummy source to that destination. For example, a penalty of $1000 in the dummy-Miami cell will prevent shortage at Miami. Of course, we cannot use this "trick" with all the destinations, because shortage must occur somewhere in the system.

The case where the supply exceeds the demand can be demonstrated by assuming that the demand at Denver is 1900 cars only. In this case, we need to add a dummy distribution center to "receive" the surplus supply. Again, the unit transportation costs to the dummy distribution center are zero, unless we require a factory to "ship out" completely. In this case, we must assign a high unit transportation cost from the designated factory to the dummy destination.

[1]To use TORA, from Main Menu select Transportation Model. From the SOLVE/MODIFY menu, select Solve ⇒ Final solution to obtain a summary of the optimum solution. A detailed description of the iterative solution of the transportation model is given in Section 5.3.3.

TABLE 5.4 MG Model with Dummy Plant

	Denver	Miami	Supply
Los Angeles	80 **1000**	215	**1000**
Detroit	100 **1300**	108	**1300**
New Orleans	102	68 **1200**	**1200**
Dummy Plant	0	0 **200**	**200**
Demand	**2300**	**1400**	

TABLE 5.5 MG Model with Dummy Destination

	Denver	Miami	Dummy	
Los Angeles	80 **1000**	215	0	**1000**
Detroit	100 **900**	108 **200**	0 **400**	**1500**
New Orleans	102	68 **1200**	0	**1200**
Demand	**1900**	**1400**	**400**	

Table 5.5 gives the new model and its optimal solution (obtained by TORA). The solution shows that the Detroit plant will have a surplus of 400 cars.

PROBLEM SET 5.1A[2]

1. True or False?
   (a) To balance a transportation model, it may be necessary to add both a dummy source and a dummy destination.
   (b) The amounts shipped to a dummy destination represent surplus at the shipping source.
   (c) The amounts shipped from a dummy source represent shortages at the receiving destinations.

[2]In this set, you may use TORA to find the optimum solution. AMPL and Solver models for the transportation problem will be introduced at the end of Section 5.3.2.

2. In each of the following cases, determine whether a dummy source or a dummy destination must be added to balance the model.
   (a) Supply: $a_1 = 10, a_2 = 5, a_3 = 4, a_4 = 6$
   Demand: $b_1 = 10, b_2 = 5, b_3 = 7, b_4 = 9$
   (b) Supply: $a_1 = 30, a_2 = 44$
   Demand: $b_1 = 25, b_2 = 30, b_3 = 10$
3. In Table 5.4 of Example 5.1-2, where a dummy plant is added, what does the solution mean when the dummy plant "ships" 150 cars to Denver and 50 cars to Miami?

*4. In Table 5.5 of Example 5.1-2, where a dummy destination is added, suppose that the Detroit plant must ship out *all* its production. How can this restriction be implemented in the model?

5. In Example 5.1-2, suppose that for the case where the demand exceeds the supply (Table 5.4), a penalty is levied at the rate of \$200 and \$300 for each undelivered car at Denver and Miami, respectively. Additionally, no deliveries are made from the Los Angeles plant to the Miami distribution center. Set up the model, and determine the optimal shipping schedule for the problem.

*6. Three electric power plants with capacities of 25, 40, and 30 million kWh supply electricity to three cities. The maximum demands at the three cities are estimated at 30, 35, and 25 million kWh. The price per million kWh at the three cities is given in Table 5.6.

   During the month of August, there is a 20% increase in demand at each of the three cities, which can be met by purchasing electricity from another network at a premium rate of \$1000 per million kWh. The network is not linked to city 3, however. The utility company wishes to determine the most economical plan for the distribution and purchase of additional energy.
   (a) Formulate the problem as a transportation model.
   (b) Determine an optimal distribution plan for the utility company.
   (c) Determine the cost of the additional power purchased by each of the three cities.

7. Solve Problem 6, assuming that there is a 10% power transmission loss through the network.

8. Three refineries with daily capacities of 6, 5, and 8 million gallons, respectively, supply three distribution areas with daily demands of 4, 8, and 7 million gallons, respectively. Gasoline is transported to the three distribution areas through a network of pipelines. The transportation cost is 10 cents per 1000 gallons per pipeline mile. Table 5.7 gives the mileage between the refineries and the distribution areas. Refinery 1 is not connected to distribution area 3.
   (a) Construct the associated transportation model.
   (b) Determine the optimum shipping schedule in the network.

TABLE 5.6 Price/Million kWh for Problem 6

	City 1	City 2	City 3
Plant 1	\$600	\$700	\$400
Plant 2	\$320	\$300	\$350
Plant 3	\$500	\$480	\$450

TABLE 5.7 Mileage Chart for Problem 8

	Distribution area 1	2	3
Refinery 1	120	180	—
Refinery 2	300	100	80
Refinery 3	200	250	120

*9. In Problem 8, suppose that the capacity of refinery 3 is 6 million gallons only and that distribution area 1 must receive all its demand. Additionally, any shortages at areas 2 and 3 will incur a penalty of 5 cents per gallon.

(a) Formulate the problem as a transportation model.

(b) Determine the optimum shipping schedule.

10. In Problem 8, suppose that the daily demand at area 3 drops to 4 million gallons. Surplus production at refineries 1 and 2 is diverted to other distribution areas by truck. The transportation cost per 100 gallons is $1.50 from refinery 1 and $2.20 from refinery 2. Refinery 3 can divert its surplus production to other chemical processes within the plant.

(a) Formulate the problem as a transportation model.

(b) Determine the optimum shipping schedule.

11. Three orchards supply crates of oranges to four retailers. The daily demand amounts at the four retailers are 150, 150, 400, and 100 crates, respectively. Supplies at the three orchards are dictated by available regular labor and are estimated at 150, 200, and 250 crates daily. However, both orchards 1 and 2 have indicated that they could supply more crates, if necessary, by using overtime labor. Orchard 3 does not offer this option. The transportation costs per crate from the orchards to the retailers are given in Table 5.8.

(a) Formulate the problem as a transportation model.

(b) Solve the problem.

(c) How many crates should orchards 1 and 2 supply using overtime labor?

12. Cars are shipped from three distribution centers to five dealers. The shipping cost is based on the mileage between the sources and the destinations, and is independent of whether the truck makes the trip with partial or full loads. Table 5.9 summarizes the mileage between the distribution centers and the dealers together with the monthly supply and demand figures given in *number* of cars. A full truckload includes 18 cars. The transportation cost per truck mile is $25.

(a) Formulate the associated transportation model.

(b) Determine the optimal shipping schedule.

TABLE 5.8 Transportation Cost/Crate for Problem 11

	Retailer 1	2	3	4
Orchard 1	$1	$2	$3	$2
Orchard 2	$2	$4	$1	$2
Orchard 3	$1	$3	$5	$3

5.2

TABLE 5.9 Mileage Chart and Supply and Demand for Problem 12

	Dealer 1	2	3	4	5	Supply
Center 1	100	150	200	140	35	**400**
Center 2	50	70	60	65	80	**200**
Center 3	40	90	100	150	130	**150**
Demand	**100**	**200**	**150**	**160**	**140**	

13. MG Auto, of Example 5.1-1, produces four car models: $M1$, $M2$, $M3$, and $M4$. The Detroit plant produces models $M1$, $M2$, and $M4$. Models $M1$ and $M2$ are also produced in New Orleans. The Los Angeles plant manufactures models $M3$ and $M4$. The capacities of the various plants and the demands at the distribution centers are given in Table 5.10.

The mileage chart is the same as given in Example 5.1-1, and the transportation rate remains at 8 cents per car mile for all models. Additionally, it is possible to satisfy a percentage of the demand for some models from the supply of others according to the specifications in Table 5.11.

(a) Formulate the corresponding transportation model.

(b) Determine the optimum shipping schedule.

(Hint: Add four new destinations corresponding to the new combinations [$M1$, $M2$], [$M3$, $M4$], [$M1$, $M2$], and [$M2$, $M4$]. The demands at the new destinations are determined from the given percentages.)

TABLE 5.10 Capacities and Demands for Problem 13

	Model $M1$	$M2$	$M3$	$M4$	Totals
Plant					
Los Angeles	—	—	700	300	1000
Detroit	500	600	—	400	1500
New Orleans	800	400	—	—	1200
Distribution center					
Denver	700	500	500	600	2300
Miami	600	500	200	100	1400

TABLE 5.11 Interchangeable Models in Problem 13

Distribution center	Percentage of demand	Interchangeable models
Denver	10	$M1, M2$
	20	$M3, M4$
Miami	10	$M1, M2$
	5	$M2, M4$

5.2 NONTRADITIONAL TRANSPORTATION MODELS

The application of the transportation model is not limited to *transporting* commodities between geographical sources and destinations. This section presents two applications in the areas of production-inventory control and tool sharpening service.

Example 5.2-1 (Production-Inventory Control)

Boralis manufactures backpacks for serious hikers. The demand for its product occurs during March to June of each year. Boralis estimates the demand for the four months to be 100, 200, 180, and 300 units, respectively. The company uses part-time labor to manufacture the backpacks and, accordingly, its production capacity varies monthly. It is estimated that Boralis can produce 50, 180, 280, and 270 units in March through June. Because the production capacity and demand for the different months do not match, a current month's demand may be satisfied in one of three ways.

1. Current month's production.
2. Surplus production in an earlier month.
3. Surplus production in a later month (backordering).

In the first case, the production cost per backpack is $40. The second case incurs an additional holding cost of $.50 per backpack per month. In the third case, an additional penalty cost of $2.00 per backpack is incurred for each month delay. Boralis wishes to determine the optimal production schedule for the four months.

The situation can be modeled as a transportation model by recognizing the following parallels between the elements of the production-inventory problem and the transportation model:

Transportation	Production-inventory
1. Source i	1. Production period i
2. Destination j	2. Demand period j
3. Supply amount at source i	3. Production capacity of period i
4. Demand at destination j	4. Demand for period j
5. Unit transportation cost from source i to destination j	5. Unit cost (production + inventory + penalty) in period i for period j

The resulting transportation model is given in Table 5.12.

TABLE 5.12 Transportation Model for Example 5.2-1

	1	2	3	4	Capacity
1	$40.00	$40.50	$41.00	$41.50	**50**
2	$42.00	$40.00	$40.50	$41.00	**180**
3	$44.00	$42.00	$40.00	$40.50	**280**
4	$46.00	$44.00	$42.00	$40.00	**270**
Demand	**100**	**200**	**180**	**300**	

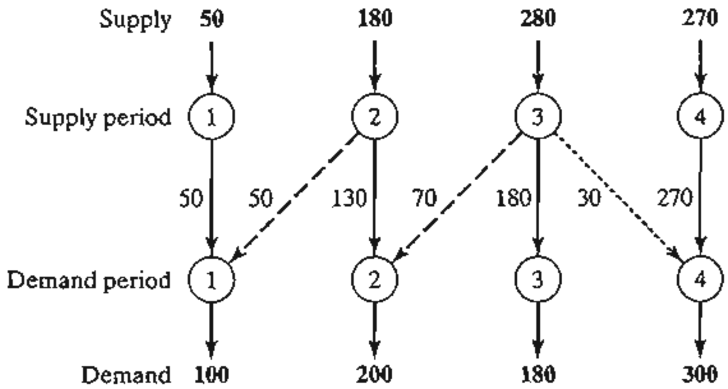


FIGURE 5.3
Optimal solution of the production-inventory model

The unit "transportation" cost from period i to period j is computed as

$$c_{ij} = \begin{cases} \text{Production cost in } i, i = j \\ \text{Production cost in } i + \text{holding cost from } i \text{ to } j, i < j \\ \text{Production cost in } i + \text{penaty cost from } i \text{ to } j, i > j \end{cases}$$

For example,

$$c_{11} = \$40.00$$

$$c_{24} = \$40.00 + (\$.50 + \$.50) = \$41.00$$

$$c_{41} = \$40.00 + (\$2.00 + \$2.00 + \$2.00) = \$46.00$$

The optimal solution is summarized in Figure 5.3. The dashed lines indicate back-ordering, the dotted lines indicate production for a future period, and the solid lines show production in a period for itself. The total cost is $31,455.

Example 5.2-2 (Tool Sharpening)

Arkansas Pacific operates a medium-sized saw mill. The mill prepares different types of wood that range from soft pine to hard oak according to a weekly schedule. Depending on the type of wood being milled, the demand for sharp blades varies from day to day according to the following 1-week (7-day) data:

Day	Mon.	Tue.	Wed.	Thu.	Fri.	Sat.	Sun.
Demand (blades)	24	12	14	20	18	14	22

The mill can satisfy the daily demand in the following manner:

1. Buy new blades at the cost of $12 a blade.
2. Use an overnight sharpening service at the cost of $6 a blade.
3. Use a slow 2-day sharpening service at the cost of $3 a blade.

The situation can be represented as a transportation model with eight sources and seven destinations. The destinations represent the 7 days of the week. The sources of the model are defined as follows: Source 1 corresponds to buying new blades, which, in the extreme case, can provide sufficient supply to cover the demand for all 7 days (= 24 + 12 + 14 + 20 + 18 + 14 + 22 = 124). Sources 2 to 8 correspond to the 7 days of the week. The amount of supply for each of these sources equals the number of used blades at the end of the associated day. For example, source 2 (i.e., Monday) will have a supply of used blades equal to the demand for Monday. The unit "transportation cost" for the model is $12, $6, or $3, depending on whether the blade is supplied from new blades, overnight sharpening, or 2-day sharpening. Notice that the overnight service means that used blades sent at the *end* of day i will be available for use at the *start* of day $i + 1$ or day $i + 2$, because the slow 2-day service will not be available until the *start* of day $i + 3$. The "disposal" column is a dummy destination needed to balance the model. The complete model and its solution are given in Table 5.13.

TABLE 5.13 Tool Sharpening Problem Expressed as a Transportation Model

	1 Mon.	2 Tue.	3 Wed.	4 Thu.	5 Fri.	6 Sat.	7 Sun.	8 Disposal	
1-New	$12 **24**	$12 **2**	$12	$12	$12	$12	$12	$0 **98**	124
2-Mon.	*M*	$6 **10**	$6 **8**	$3 **6**	$3	$3	$3	$0	24
3-Tue.	*M*	*M*	$6 **6**	$6	$3 **6**	$3	$3	$0	12
4-Wed.	*M*	*M*	*M*	$6 **14**	$6	$3	$3	$0	14
5-Thu.	*M*	*M*	*M*	*M*	$6 **12**	$6	$3 **8**	$0	20
6-Fri.	*M*	*M*	*M*	*M*	*M*	$6 **14**	$6	$0 **4**	18
7-Sat.	*M*	*M*	*M*	*M*	*M*	*M*	$6 **14**	$0	14
8-Sun.	*M*	*M*	*M*	*M*	*M*	*M*	*M*	$0 **22**	22
	24	12	14	20	18	14	22	124	

The problem has alternative optima at a cost of \$840 (file toraEx5.2-2.txt). The following table summarizes one such solution.

	Number of sharp blades (Target day)			
Period	*New*	*Overnight*	*2-day*	Disposal
Mon.	24 (Mon.)	10 (Tue.) + 8 (Wed.)	6 (Thu.)	0
Tues.	2 (Tue.)	6 (Wed.)	6 (Fri.)	0
Wed.	0	14 (Thu.)	0	0
Thu.	0	12 (Fri.)	8 (Sun.)	0
Fri.	0	14 (Sat.)	0	4
Sat.	0	14 (Sun.)	0	0
Sun.	0	0	0	22

Remarks. The model in Table 5.13 is suitable only for the first week of operation because it does not take into account the *rotational* nature of the days of the week, in the sense that this week's days can act as sources for next week's demand. One way to handle this situation is to assume that the very first week of operation starts with all new blades for each day. From then on, we use a model consisting of exactly 7 sources and 7 destinations corresponding to the days of the week. The new model will be similar to Table 5.13 less source "New" and destination "Disposal." Also, only diagonal cells will be blocked (unit cost $= M$). The remaining cells will have a unit cost of either \$3.00 or \$6.00. For example, the unit cost for cell (Sat., Mon.) is \$6.00 and that for cells (Sat., Tue.), (Sat., Wed.), (Sat., Thu.), and (Sat., Fri.) is \$3.00. The table below gives the solution costing \$372. As expected, the optimum solution will always use the 2-day service only. The problem has alternative optima (see file toraEx5.2-2a.txt).

	Week $i+1$							
Week i	*Mon.*	*Tue.*	*Wed.*	*Thu.*	*Fri.*	*Sat.*	*Sun.*	Total
Mon.				6			18	24
Tue.					8		4	12
Wed.	12					2		14
Thu.	8	12						20
Fri.	4		14					18
Sat.				14				14
Sun.					10	12		22
Total	24	12	14	20	18	14	22	

PROBLEM SET 5.2A[3]

1. In Example 5.2-1, suppose that the holding cost per unit is period-dependent and is given by 40, 30, and 70 cents for periods 1, 2, and 3, respectively. The penalty and production costs remain as given in the example. Determine the optimum solution and interpret the results.

[3] In this set, you may use TORA to find the optimum solution. AMPL and Solver models for the transportation problem will be introduced at the end of Section 5.3.2.

*2. In Example 5.2-2, suppose that the sharpening service offers 3-day service for $1 a blade on Monday and Tuesday (days 1 and 2). Reformulate the problem, and interpret the optimum solution.

3. In Example 5.2-2, if a blade is not used the day it is sharpened, a holding cost of 50 cents per blade per day is incurred. Reformulate the model, and interpret the optimum solution.

4. JoShop wants to assign four different categories of machines to five types of tasks. The numbers of machines available in the four categories are 25, 30, 20, and 30. The numbers of jobs in the five tasks are 20, 20, 30, 10, and 25. Machine category 4 cannot be assigned to task type 4. Table 5.14 provides the unit cost (in dollars) of assigning a machine category to a task type. The objective of the problem is to determine the optimum number of machines in each category to be assigned to each task type. Solve the problem and interpret the solution.

*5. The demand for a perishable item over the next four months is 400, 300, 420, and 380 tons, respectively. The supply capacities for the same months are 500, 600, 200, and 300 tons. The purchase price per ton varies from month to month and is estimated at $100, $140, $120, and $150, respectively. Because the item is perishable, a current month's supply must be consumed within 3 months (starting with current month). The storage cost per ton per month is $3. The nature of the item does not allow back-ordering. Solve the problem as a transportation model and determine the optimum delivery schedule for the item over the next 4 months.

6. The demand for a special small engine over the next five quarters is 200, 150, 300, 250, and 400 units. The manufacturer supplying the engine has different production capacities estimated at 180, 230, 430, 300, and 300 for the five quarters. Back-ordering is not allowed, but the manufacturer may use overtime to fill the immediate demand, if necessary. The overtime capacity for each period is half the regular capacity. The production costs per unit for the five periods are $100, $96, $116, $102, and $106, respectively. The overtime production cost per engine is 50% higher than the regular production cost. If an engine is produced now for use in later periods, an additional storage cost of $4 per engine per period is incurred. Formulate the problem as a transportation model. Determine the optimum number of engines to be produced during regular time and overtime of each period.

7. Periodic preventive maintenance is carried out on aircraft engines, where an important component must be replaced. The numbers of aircraft scheduled for such maintenance over the next six months are estimated at 200, 180, 300, 198, 230, and 290, respectively. All maintenance work is done during the first day of the month, where a used component may be replaced with a new or an overhauled component. The overhauling of used components may be done in a local repair facility, where they will be ready for use at the beginning of next month, or they may be sent to a central repair shop, where a delay of

TABLE 5.14 Unit Costs for Problem 4

		Task type				
		1	2	3	4	5
Machine category	1	10	2	3	15	9
	2	5	10	15	2	4
	3	15	5	14	7	15
	4	20	15	13	—	8

TABLE 5.15 Bids per Acre for Problem 8

		Location 1	Location 2	Location 3
Bidder	1	\$520	\$210	\$570
	2	—	\$510	\$495
	3	\$650	—	\$240
	4	\$180	\$430	\$710

3 months (including the month in which maintenance occurs) is expected. The repair cost in the local shop is \$120 per component. At the central facility, the cost is only \$35 per component. An overhauled component used in a later month will incur an additional storage cost of \$1.50 per unit per month. New components may be purchased at \$200 each in month 1, with a 5% price increase every 2 months. Formulate the problem as a transportation model, and determine the optimal schedule for satisfying the demand for the component over the next six months.

8. The National Parks Service is receiving four bids for logging at three pine forests in Arkansas. The three locations include 10,000, 20,000, and 30,000 acres. A single bidder can bid for at most 50% of the total acreage available. The bids per acre at the three locations are given in Table 5.15. Bidder 2 does not wish to bid on location 1, and bidder 3 cannot bid on location 2.

(a) In the present situation, we need to *maximize* the total bidding revenue for the Parks Service. Show how the problem can be formulated as a transportation model.

(b) Determine the acreage that should be assigned to each of the four bidders.

5.3 THE TRANSPORTATION ALGORITHM

The transportation algorithm follows the *exact steps* of the simplex method (Chapter 3). However, instead of using the regular simplex tableau, we take advantage of the special structure of the transportation model to organize the computations in a more convenient form.

The special transportation algorithm was developed early on when hand computations were the norm and the shortcuts were warranted. Today, we have powerful computer codes that can solve a transportation model of any size as a regular LP.[4] Nevertheless, the transportation algorithm, aside from its historical significance, does provide insight into the use of the theoretical primal-dual relationships (introduced in Section 4.2) to achieve a practical end result, that of improving hand computations. The exercise is theoretically intriguing.

The details of the algorithm are explained using the following numeric example.

[4]In fact, TORA handles all necessary computations in the background using the regular simplex method and uses the transportation model format only as a screen "veneer."

TABLE 5.16 SunRay Transportation Model

	Mill 1	2	3	4	Supply
Silo 1	10 x_{11}	2 x_{12}	20 x_{13}	11 x_{14}	15
2	12 x_{21}	7 x_{22}	9 x_{23}	20 x_{24}	25
3	4 x_{31}	14 x_{32}	16 x_{33}	18 x_{34}	10
Demand	5	15	15	15	

Example 5.3-1 (SunRay Transport)

SunRay Transport Company ships truckloads of grain from three silos to four mills. The supply (in truckloads) and the demand (also in truckloads) together with the unit transportation costs per truckload on the different routes are summarized in the transportation model in Table 5.16. The unit transportation costs, c_{ij}, (shown in the northeast corner of each box) are in hundreds of dollars. The model seeks the minimum-cost shipping schedule x_{ij} between silo i and mill j ($i = 1, 2, 3; j = 1, 2, 3, 4$).

Summary of the Transportation Algorithm. The steps of the transportation algorithm are exact parallels of the simplex algorithm.

Step 1. Determine a *starting* basic feasible solution, and go to step 2.

Step 2. Use the optimality condition of the simplex method to determine the *entering variable* from among all the nonbasic variables. If the optimality condition is satisfied, stop. Otherwise, go to step 3.

Step 3. Use the feasibility condition of the simplex method to determine the *leaving variable* from among all the current basic variables, and find the new basic solution. Return to step 2.

5.3.1 Determination of the Starting Solution

A general transportation model with m sources and n destinations has $m + n$ constraint equations, one for each source and each destination. However, because the transportation model is always balanced (sum of the supply = sum of the demand), one of these equations is redundant. Thus, the model has $m + n - 1$ independent constraint equations, which means that the starting basic solution consists of $m + n - 1$ basic variables. Thus, in Example 5.3-1, the starting solution has $3 + 4 - 1 = 6$ basic variables.

The special structure of the transportation problem allows securing a nonartificial starting basic solution using one of three methods:[5]

1. Northwest-corner method
2. Least-cost method
3. Vogel approximation method

The three methods differ in the "quality" of the starting basic solution they produce, in the sense that a better starting solution yields a smaller objective value. In general, though not always, the Vogel method yields the best starting basic solution, and the northwest-corner method yields the worst. The tradeoff is that the northwest-corner method involves the least amount of computations.

Northwest-Corner Method. The method starts at the northwest-corner cell (route) of the tableau (variable x_{11}).

Step 1. Allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount.

Step 2. Cross out the row or column with zero supply or demand to indicate that no further assignments can be made in that row or column. If both a row and a column net to zero simultaneously, *cross out one only*, and leave a zero supply (demand) in the uncrossed-out row (column).

Step 3. If *exactly one* row or column is left uncrossed out, stop. Otherwise, move to the cell to the right if a column has just been crossed out or below if a row has been crossed out. Go to step 1.

Example 5.3-2

The application of the procedure to the model of Example 5.3-1 gives the starting basic solution in Table 5.17. The arrows show the order in which the allocated amounts are generated.

The starting basic solution is

$$x_{11} = 5, x_{12} = 10$$

$$x_{22} = 5, x_{23} = 15, x_{24} = 5$$

$$x_{34} = 10$$

The associated cost of the schedule is

$$z = 5 \times 10 + 10 \times 2 + 5 \times 7 + 15 \times 9 + 5 \times 20 + 10 \times 18 = \$520$$

Least-Cost Method. The least-cost method finds a better starting solution by concentrating on the cheapest routes. The method assigns as much as possible to the cell with the smallest unit cost (ties are broken arbitrarily). Next, the satisfied row or column is crossed out and the amounts of supply and demand are adjusted accordingly.

[5]All three methods are featured in TORA's tutorial module. See the end of Section 5.3.3.

TABLE 5.17 Northwest-Corner Starting Solution

	1	2	3	4	Supply
1	10 5	2 10	20	11	15
2	12	7 5	9 15	20 5	25
3	4	14	16	18 10	10
Demand	5	15	15	15	

If both a row and a column are satisfied simultaneously, *only one is crossed out*, the same as in the northwest-corner method. Next, look for the uncrossed-out cell with the smallest unit cost and repeat the process until exactly one row or column is left uncrossed out.

Example 5.3-3

The least-cost method is applied to Example 5.3-1 in the following manner:

1. Cell (1, 2) has the least unit cost in the tableau (= \$2). The most that can be shipped through (1,2) is $x_{12} = 15$ truckloads, which happens to satisfy both row 1 and column 2 simultaneously. We arbitrarily cross out column 2 and adjust the supply in row 1 to 0.
2. Cell (3, 1) has the smallest uncrossed-out unit cost (= \$4). Assign $x_{31} = 5$, and cross out column 1 because it is satisfied, and adjust the demand of row 3 to $10 - 5 = 5$ truckloads.
3. Continuing in the same manner, we successively assign 15 truckloads to cell (2, 3), 0 truckloads to cell (1, 4), 5 truckloads to cell (3, 4), and 10 truckloads to cell (2, 4) (verify!).

The resulting starting solution is summarized in Table 5.18. The arrows show the order in which the allocations are made. The starting solution (consisting of 6 basic variables) is $x_{12} = 15, x_{14} = 0, x_{23} = 15, x_{24} = 10, x_{31} = 5, x_{34} = 5$. The associated objective value is

$$z = 15 \times 2 + 0 \times 11 + 15 \times 9 + 10 \times 20 + 5 \times 4 + 5 \times 18 = \$475$$

The quality of the least-cost starting solution is better than that of the northwest-corner method (Example 5.3-2) because it yields a smaller value of z (\$475 versus \$520 in the northwest-corner method).

Vogel Approximation Method (VAM). VAM is an improved version of the least-cost method that generally, but not always, produces better starting solutions.

Step 1. For each row (column), determine a penalty measure by subtracting the *smallest* unit cost element in the row (column) from the *next smallest* unit cost element in the same row (column).

TABLE 5.18 Least-Cost Starting Solution

	1	2	3	4	Supply
1	10	(start) 2 15	20	11 0	15
2	12	7	9 15	(end) 20 10	25
3	4 5	14	16	18 5	10
Demand	5	15	15	15	

Step 2. Identify the row or column with the largest penalty. Break ties arbitrarily. Allocate as much as possible to the variable with the least unit cost in the selected row or column. Adjust the supply and demand, and cross out the satisfied row *or* column. If a row and a column are satisfied simultaneously, only one of the two is crossed out, and the remaining row (column) is assigned zero supply (demand).

Step 3. **(a)** If exactly one row or column with zero supply or demand remains uncrossed out, stop.

(b) If one row (column) with *positive* supply (demand) remains uncrossed out, determine the basic variables in the row (column) by the least-cost method. Stop.

(c) If all the uncrossed out rows and columns have (remaining) zero supply and demand, determine the *zero* basic variables by the least-cost method. Stop.

(d) Otherwise, go to step 1.

Example 5.3-4

VAM is applied to Example 5.3-1. Table 5.19 computes the first set of penalties.

Because row 3 has the largest penalty (= 10) and cell (3, 1) has the smallest unit cost in that row, the amount 5 is assigned to x_{31}. Column 1 is now satisfied and must be crossed out. Next, new penalties are recomputed as in Table 5.20.

Table 5.20 shows that row 1 has the highest penalty (= 9). Hence, we assign the maximum amount possible to cell (1, 2), which yields $x_{12} = 15$ and simultaneously satisfies both row 1 and column 2. We arbitrarily cross out column 2 and adjust the supply in row 1 to zero.

Continuing in the same manner, row 2 will produce the highest penalty (= 11), and we assign $x_{23} = 15$, which crosses out column 3 and leaves 10 units in row 2. Only column 4 is left, and it has a positive supply of 15 units. Applying the least-cost method to that column, we successively assign $x_{14} = 0$, $x_{34} = 5$, and $x_{24} = 10$ (verify!). The associated objective value for this solution is

$$z = 15 \times 2 + 0 \times 11 + 15 \times 9 + 10 \times 20 + 5 \times 4 + 5 \times 18 = \$475$$

This solution happens to have the same objective value as in the least-cost method.

TABLE 5.19 Row and Column Penalties in VAM

	1	2	3	4		Row penalty
1	10	2	20	11	15	10 − 2 = 8
2	12	7	9	20	25	9 − 7 = 2
3	4 / 5	14	16	18	10	14 − 4 = 10
	5	15	15	15		
Column penalty	10 − 4 = 6	7 − 2 = 5	16 − 9 = 7	18 − 11 = 7		

TABLE 5.20 First Assignment in VAM ($x_{31} = 5$)

	1	2	3	4		Row penalty
1	10	2	20	11	15	9
2	12	7	9	20	25	2
3	4 / 5	14	16	18	10	2
	5	15	15	15		
Column penalty	—	5	7	7		

PROBLEM SET 5.3A

1. Compare the starting solutions obtained by the northwest-corner, least-cost, and Vogel methods for each of the following models:

*(a)

0	2	1	6
2	1	5	7
2	4	3	7
5	5	10	

(b)

1	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

(c)

5	1	8	12
2	4	0	14
3	6	7	4
9	10	11	

5.3.2 Iterative Computations of the Transportation Algorithm

After determining the starting solution (using any of the three methods in Section 5.3.1), we use the following algorithm to determine the optimum solution:

Step 1. Use the simplex *optimality condition* to determine the *entering variable* as the current nonbasic variable that can improve the solution. If the optimality condition is satisfied, stop. Otherwise, go to step 2.

Step 2. Determine the *leaving variable* using the simplex *feasibility condition*. Change the basis, and return to step 1.

The optimality and feasibility conditions do not involve the familiar row operations used in the simplex method. Instead, the special structure of the transportation model allows simpler computations.

Example 5.3-5

Solve the transportation model of Example 5.3-1, starting with the northwest-corner solution.

Table 5.21 gives the northwest-corner starting solution as determined in Table 5.17, Example 5.3-2.

The determination of the entering variable from among the current nonbasic variables (those that are not part of the starting basic solution) is done by computing the nonbasic coefficients in the z-row, using the **method of multipliers** (which, as we show in Section 5.3.4, is rooted in LP duality theory).

In the method of multipliers, we associate the multipliers u_i and v_j with row i and column j of the transportation tableau. For each current *basic* variable x_{ij}, these multipliers are shown in Section 5.3.4 to satisfy the following equations:

$$u_i + v_j = c_{ij}, \text{ for each } basic\ x_{ij}$$

As Table 5.21 shows, the starting solution has 6 basic variables, which leads to 6 equations in 7 unknowns. To solve these equations, the method of multipliers calls for arbitrarily setting any $u_i = 0$, and then solving for the remaining variables as shown below.

Basic variable	(u, v) Equation	Solution
x_{11}	$u_1 + v_1 = 10$	Set $u_1 = 0 \rightarrow v_1 = 10$
x_{12}	$u_1 + v_2 = 2$	$u_1 = 0 \rightarrow v_2 = 2$
x_{22}	$u_2 + v_2 = 7$	$v_2 = 2 \rightarrow u_2 = 5$
x_{23}	$u_2 + v_3 = 9$	$u_2 = 5 \rightarrow v_3 = 4$
x_{24}	$u_2 + v_4 = 20$	$u_2 = 5 \rightarrow v_4 = 15$
x_{34}	$u_3 + v_4 = 18$	$v_4 = 15 \rightarrow u_3 = 3$

To summarize, we have

$$u_1 = 0, u_2 = 5, u_3 = 3$$
$$v_1 = 10, v_2 = 2, v_3 = 4, v_4 = 15$$

Next, we use u_i and v_j to evaluate the nonbasic variables by computing

$$u_i + v_j - c_{ij}, \text{ for each } nonbasic\ x_{ij}$$

TABLE 5.21 Starting Iteration

	1	2	3	4	Supply
1	10 / **5**	2 / **10**	20	11	**15**
2	12	7 / **5**	9 / **15**	20 / **5**	**25**
3	4	14	16	18 / **10**	**10**
Demand	**5**	**15**	**15**	**15**	

The results of these evaluations are shown in the following table:

Nonbasic variable	$u_i + v_j - c_{ij}$
x_{13}	$u_1 + v_3 - c_{13} = 0 + 4 - 20 = -16$
x_{14}	$u_1 + v_4 - c_{14} = 0 + 15 - 11 = 4$
x_{21}	$u_2 + v_1 - c_{21} = 5 + 10 - 12 = 3$
x_{31}	$u_3 + v_1 - c_{31} = 3 + 10 - 4 = 9$
x_{32}	$u_3 + v_2 - c_{32} = 3 + 2 - 14 = -9$
x_{33}	$u_3 + v_3 - c_{33} = 3 + 4 - 16 = -9$

The preceding information, together with the fact that $u_i + v_j - c_{ij} = 0$ for each basic x_{ij}, is actually equivalent to computing the z-row of the simplex tableau, as the following summary shows.

Basic	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	↓ x_{31}	x_{32}	x_{33}	x_{34}
z	0	0	−16	4	3	0	0	0	9	−9	−9	0

Because the transportation model seeks to *minimize* cost, the entering variable is the one having the *most positive* coefficient in the z-row. Thus, x_{31} is the entering variable.

The preceding computations are usually done directly on the transportation tableau as shown in Table 5.22, meaning that it is not necessary really to write the (u, v)-equations explicitly. Instead, we start by setting $u_1 = 0$.[6] Then we can compute the v-values of all the columns that have *basic* variables in row 1—namely, v_1 and v_2. Next, we compute u_2 based on the (u, v)-equation of basic x_{22}. Now, given u_2, we can compute v_3 and v_4. Finally, we determine u_3 using the basic equation of x_{33}. Once all the u's and v's have been determined, we can evaluate the nonbasic variables by computing $u_i + v_j - c_{ij}$ for each nonbasic x_{ij}. These evaluations are shown in Table 5.22 in the boxed southeast corner of each cell.

Having identified x_{31} as the entering variable, we need to determine the leaving variable. Remember that if x_{31} enters the solution to become basic, one of the current basic variables must leave as nonbasic (at zero level).

TABLE 5.22 Iteration 1 Calculations

	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 \equiv 0$	10 **5**	2 **10**	20 [−16]	11 [4]	**15**
$u_2 = 5$	12 [3]	7 **5**	9 **15**	20 **5**	**25**
$u_3 = 3$	4 [9]	14 [−9]	16 [−9]	18 **10**	**10**
Demand	**5**	**15**	**15**	**15**	

[6]The tutorial module of TORA is designed to demonstrate that assigning a zero initial value to any u or v does not affect the optimization results. See TORA Moment on page 216.

The selection of x_{31} as the entering variable means that we want to ship through this route because it reduces the total shipping cost. What is the most that we can ship through the new route? Observe in Table 5.22 that if route (3, 1) ships θ units (i.e., $x_{31} = \theta$), then the maximum value of θ is determined based on two conditions.

1. Supply limits and demand requirements remain satisfied.
2. Shipments through all routes remain nonnegative.

These two conditions determine the maximum value of θ and the leaving variable in the following manner. First, construct a *closed loop* that starts and ends at the entering variable cell, (3, 1). The loop consists of *connected horizontal* and *vertical* segments only (no diagonals are allowed).[7] Except for the entering variable cell, each corner of the closed loop must coincide with a basic variable. Table 5.23 shows the loop for x_{31}. Exactly one loop exists for a given entering variable.

Next, we assign the amount θ to the entering variable cell (3, 1). For the supply and demand limits to remain satisfied, we must alternate between subtracting and adding the amount θ at the successive *corners* of the loop as shown in Table 5.23 (it is immaterial whether the loop is traced in a clockwise or counterclockwise direction). For $\theta \geq 0$, the new values of the variables then remain nonnegative if

$$x_{11} = 5 - \theta \geq 0$$

$$x_{22} = 5 - \theta \geq 0$$

$$x_{34} = 10 - \theta \geq 0$$

The corresponding maximum value of θ is 5, which occurs when both x_{11} and x_{22} reach zero level. Because only one current basic variable must leave the basic solution, we can choose either x_{11} or x_{22} as the leaving variable. We arbitrarily choose x_{11} to leave the solution.

The selection of x_{31} (= 5) as the entering variable and x_{11} as the leaving variable requires adjusting the values of the basic variables at the corners of the closed loop as Table 5.24 shows. Because each unit shipped through route (3, 1) reduces the shipping cost by \$9 (= $u_3 + v_1 - c_{31}$), the total cost associated with the new schedule is \$9 × 5 = \$45 less than in the previous schedule. Thus, the new cost is \$520 − \$45 = \$475.

TABLE 5.23 Determination of Closed Loop for x_{31}

	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 = 0$	10 5 − Θ	2 10 + Θ	20 −16	11 4	15
$u_2 = 5$	12 3	7 5 − Θ	9 15	20 5 + Θ	25
$u_3 = 3$	4 Θ 9	14 −9	16 −9	18 10 − Θ	10
Demand	5	15	15	15	

[7]TORA's tutorial module allows you to determine the cells of the *closed loop* interactively with immediate feedback regarding the validity of your selections. See TORA Moment on page 216.

TABLE 5.24 Iteration 2 Calculations

	$v_1 = 1$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 \equiv 0$	(10) −9	(2) 15 − Θ	(20) −16	(11) Θ 4	15
$u_2 = 5$	(12) −6	(7) 0 + Θ	(9) 15	(20) 10 − Θ	25
$u_3 = 3$	(4) 5	(14) −9	(16) −9	(18) 5	10
Demand	5	15	15	15	

TABLE 5.25 Iteration 3 Calculations (Optimal)

	$v_1 = -3$	$v_2 = 2$	$v_3 = 4$	$v_4 = 11$	Supply
$u_1 \equiv 0$	(10) −13	(2) 5	(20) −16	(11) 10	15
$u_2 = 5$	(12) −10	(7) 10	(9) 15	(20) −4	25
$u_3 = 7$	(4) 5	(14) −5	(16) −5	(18) 5	10
Demand	5	15	15	15	

Given the new basic solution, we repeat the computation of the multipliers u and v, as Table 5.24 shows. The entering variable is x_{14}. The closed loop shows that $x_{14} = 10$ and that the leaving variable is x_{24}.

The new solution, shown in Table 5.25, costs $4 \times 10 = \$40$ less than the preceding one, thus yielding the new cost $\$475 - \$40 = \$435$. The new $u_i + v_j - c_{ij}$ are now negative for all nonbasic x_{ij}. Thus, the solution in Table 5.25 is optimal.

The following table summarizes the optimum solution.

From silo	To mill	Number of truckloads
1	2	5
1	4	10
2	2	10
2	3	15
3	1	5
3	4	5
	Optimal cost = $435	

TORA Moment.

From Solve/Modify Menu, select Solve ⇒ Iterations, and choose one of the three methods (northwest corner, least-cost, or Vogel) to start the transportation model iterations. The iterations module offers two useful interactive features:

1. You can set any u or v to zero before generating Iteration 2 (the default is $u_1 = 0$). Observe then that although the values of u_i and v_j change, the evaluation of the nonbasic cells ($= u_i + v_j - c_{ij}$) remains the same. This means that, initially, any u or v can be set to zero (in fact, any value) without affecting the optimality calculations.
2. You can test your understanding of the selection of the *closed loop* by clicking (in any order) the *corner* cells that comprise the path. If your selection is correct, the cell will change color (green for entering variable, red for leaving variable, and gray otherwise).

Solver Moment.

Entering the transportation model into Excel spreadsheet is straightforward. Figure 5.4 provides the Excel Solver template for Example 5.3-1 (file solverEx5.3-1.xls), together with all the formulas and the definition of range names.

In the input section, data include unit cost matrix (cells B4:E6), source names (cells A4:A6), destination names (cells B3:E3), supply (cells F4:F6), and demand (cells B7:E7). In the output section, cells B11:E13 provide the optimal solution in matrix form. The total cost formula is given in target cell A10.

AMPL Moment.

Figure 5.5 provides the AMPL model for the transportation model of Example 5.3-1 (file amplEx5.3-1a.txt). The names used in the model are self-explanatory. Both the constraints and the objective function follow the format of the LP model presented in Example 5.1-1.

The model uses the sets `sNodes` and `dNodes` to conveniently allow the use of the alphanumeric set members `{S1, S2, S3}` and `{D1, D2, D3, D4}` which are entered in the data section. All the input data are then entered in terms of these set members as shown in Figure 5.5.

Although the alphanumeric code for set members is more readable, generating them for large problems may not be convenient. File amplEx5.3-1b shows how the same sets can be defined as `{1..m}` and `{1..n}`, where `m` and `n` represent the number of sources and the number of destinations. By simply assigning numeric values for `m` and `n`, the sets are automatically defined for any size model.

The data of the transportation model can be retrieved from a spreadsheet (file TM.xls) using the AMPL `table` statement. File amplEx3.5-1c.txt provides the details. To study this model, you will need to review the material in Section A.5.5.

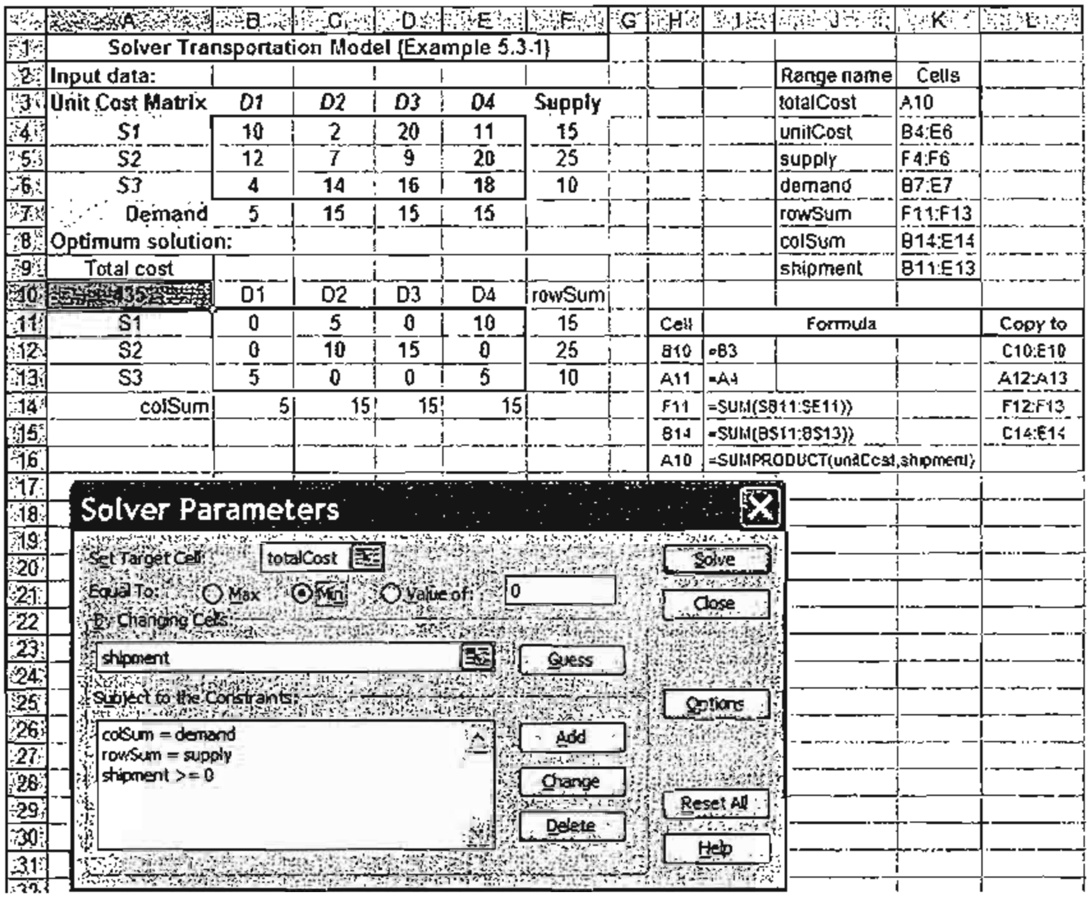


FIGURE 5.4

Excel Solver solution of the transportation model of Example 5.3-1 (File solverEx5.3-1.xls)

PROBLEM SET 5.3B

1. Consider the transportation models in Table 5.26.
   - **(a)** Use the northwest-corner method to find the starting solution.
   - **(b)** Develop the iterations that lead to the optimum solution.
   - **(c)** *TORA Experiment.* Use TORA's Iterations module to compare the effect of using the northwest-corner rule, least-cost method, and Vogel method on the number of iterations leading to the optimum solution.
   - **(d)** *Solver Experiment.* Solve the problem by modifying file solverEx5.3-1.xls.
   - **(e)** *AMPL Experiment.* Solve the problem by modifying file amplEx5.3-1b.txt.
2. In the transportation problem in Table 5.27, the total demand exceeds the total supply. Suppose that the penalty costs per unit of unsatisfied demand are \$5, \$3, and \$2 for destinations 1, 2, and 3, respectively. Use the least-cost starting solution and compute the iterations leading to the optimum solution.

```
#----- Transporation model (Example 5.3-1)-----
set sNodes;
set dNodes;
param c{sNodes,dNodes};
param supply{sNodes};
param demand{dNodes};
var x{sNodes,dNodes}>=0;
minimize z:sum {i in sNodes,j in dNodes}c[i,j]*x[i,j];
subject to
source {i in sNodes}:sum{j in dNodes}x[i,j]=supply[i];
dest {j in dNodes}:sum{i in sNodes}x[i,j]=demand[j];
data;
set sNodes:=S1 S2 S3;
set dNodes:=D1 D2 D3 D4;
param c:
   D1  D2  D3  D4 :=
S1 10  2   20  11
S2 12  7   9   20
S3 4   14  16  18;
param supply:= S1 15 S2 25 S3 10;
param demand:=D1 5 D2 15 D3 15 D4 15;
solve;display z, x;
```

FIGURE 5.5

AMPL model of the transportation model of Example 5.3-1 (File amplEx5.3-1a.txt)

TABLE 5.26 Transportation Models for Problem 1

(i)

$0	$2	$1	6
$2	$1	$5	9
$2	$4	$3	5
5	**5**	**10**	

(ii)

$10	$4	$2	**8**
$2	$3	$4	**5**
$1	$2	$0	**6**
7	**6**	**6**	

(iii)

—	$3	$5	**4**
$7	$4	$9	**7**
$1	$8	$6	**19**
5	**6**	**19**	

TABLE 5.27 Data for Problem 2

$5	$1	$7	**10**
$6	$4	$6	**80**
$3	$2	$5	**15**
75	**20**	**50**	

3. In Problem 2, suppose that there are no penalty costs, but that the demand at destination 3 must be satisfied completely.

(a) Find the optimal solution.

(b) *Solver Experiment.* Solve the problem by modifying file solverEx5.3-1.xls.

(c) *AMPL Experiment.* Solve the problem by modifying file amplEx5.3b-1.txt.

TABLE 5.28 Data for Problem 4

\$1	\$2	\$1	20
\$3	\$4	\$5	40
\$2	\$3	\$3	30
30	20	20	

TABLE 5.29 Data for Problem 6

10			10
	20	20	40
10	20	20	

4. In the unbalanced transportation problem in Table 5.28, if a unit from a source is not shipped out (to any of the destinations), a storage cost is incurred at the rate of \$5, \$4, and \$3 per unit for sources 1, 2, and 3, respectively. Additionally, all the supply at source 2 must be shipped out completely to make room for a new product. Use Vogel's starting solution and determine all the iterations leading to the optimum shipping schedule.

*5. In a 3 $\times$ 3 transportation problem, let x_{ij} be the amount shipped from source i to destination j and let c_{ij} be the corresponding transportation cost per unit. The amounts of supply at sources 1, 2, and 3 are 15, 30, and 85 units, respectively, and the demands at destinations 1, 2, and 3 are 20, 30, and 80 units, respectively. Assume that the starting northwest-corner solution is optimal and that the associated values of the multipliers are given as $u_1 = -2, u_2 = 3, u_3 = 5, v_1 = 2, v_2 = 5$, and $v_3 = 10$.
   (a) Find the associated optimal cost.
   (b) Determine the smallest value of c_{ij} for each nonbasic variable that will maintain the optimality of the northwest-corner solution.

6. The transportation problem in Table 5.29 gives the indicated *degenerate* basic solution (i.e., at least one of the basic variables is zero). Suppose that the multipliers associated with this solution are $u_1 = 1, u_2 = -1, v_1 = 2, v_2 = 2$, and $v_3 = 5$ and that the unit cost for all (basic and nonbasic) *zero* x_{ij} variables is given by

$$c_{ij} = i + j\theta, -\infty < \theta < \infty$$

   (a) If the given solution is optimal, determine the associated optimal value of the objective function.
   (b) Determine the value of θ that will guarantee the optimality of the given solution. (*Hint:* Locate the zero basic variable.)

7. Consider the problem

$$\text{Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$$

on

TABLE 5.30 Data for Problem 7

\$1	\$1	\$2	5
\$6	\$5	\$1	6
2	7	1	

subject to

$$\sum_{j=1}^{n} x_{ij} \geq a_i, i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j, j = 1, 2, \ldots, n$$

$$x_{ij} \geq 0, \text{ all } i \text{ and } j$$

It may appear logical to assume that the optimum solution will require the first (second) set of inequalities to be replaced with equations if $\sum a_i \geq \sum b_j$ ($\sum a_i \leq \sum b_j$). The counterexample in Table 5.30 shows that this assumption is not correct.

Show that the application of the suggested procedure yields the solution $x_{11} = 2$, $x_{12} = 3$, $x_{22} = 4$, and $x_{23} = 2$, with $z = \$27$, which is worse than the feasible solution $x_{11} = 2$, $x_{12} = 7$, and $x_{23} = 6$, with $z = \$15$.

5.3.3 Simplex Method Explanation of the Method of Multipliers

The relationship between the method of multipliers and the simplex method can be explained based on the primal-dual relationships (Section 4.2). From the special structure of the LP representing the transportation model (see Example 5.1-1 for an illustration), the associated dual problem can be written as

$$\text{Maximize } z = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$$

subject to

$$u_i + v_j \leq c_{ij}, \text{ all } i \text{ and } j$$

$$u_i \text{ and } v_j \text{ unrestricted}$$

where

a_i = Supply amount at source i

b_j = Demand amount at destination j

c_{ij} = Unit transportation cost from source i to destination j

u_i = Dual variable of the constraint associated with source i

v_j = Dual variable of the constraint associated with destination j

From Formula 2, Section 4.2.4, the objective-function coefficients (reduced costs) of the variable x_{ij} equal the difference between the left- and right-hand sides of the corresponding dual constraint—that is, $u_i + v_j - c_{ij}$. However, we know that this quantity must equal zero for each *basic variable*, which then produces the following result:

$$u_i + v_j = c_{ij}, \text{ for each } \textit{basic} \text{ variable } x_{ij}.$$

There are $m + n - 1$ such equations whose solution (after assuming an arbitrary value $u_1 = 0$) yields the multipliers u_i and v_j. Once these multipliers are computed, the entering variable is determined from all the *nonbasic* variables as the one having the largest positive $u_i + v_j - c_{ij}$.

The assignment of an arbitrary value to one of the dual variables (i.e., $u_1 = 0$) may appear inconsistent with the way the dual variables are computed using Method 2 in Section 4.2.3. Namely, for a given basic solution (and, hence, inverse), the dual values must be unique. Problem 2, Set 5.3c, addresses this point.

PROBLEM SET 5.3C

1. Write the dual problem for the LP of the transportation problem in Example 5.3-5 (Table 5.21). Compute the associated optimum *dual* objective value using the optimal dual values given in Table 5.25, and show that it equals the optimal cost given in the example.
2. In the transportation model, one of the dual variables assumes an arbitrary value. This means that for the same basic solution, the values of the associated dual variables are not unique. The result appears to contradict the theory of linear programming, where the dual values are determined as the product of the vector of the objective coefficients for the basic variables and the associated inverse basic matrix (see Method 2, Section 4.2.3). Show that for the transportation model, although the inverse basis is unique, the vector of *basic* objective coefficients need not be so. Specifically, show that if c_{ij} is changed to $c_{ij} + k$ for all i and j, where k is a constant, then the optimal values of x_{ij} will remain the same. Hence, the use of an arbitrary value for a dual variable is implicitly equivalent to assuming that a specific constant k is added to all c_{ij}.

5.4 THE ASSIGNMENT MODEL

"The best person for the job" is an apt description of the assignment model. The situation can be illustrated by the assignment of workers with varying degrees of skill to jobs. A job that happens to match a worker's skill costs less than one in which the operator is not as skillful. The objective of the model is to determine the minimum-cost assignment of workers to jobs.

The general assignment model with n workers and n jobs is represented in Table 5.31.

The element c_{ij} represents the cost of assigning worker i to job j ($i, j = 1, 2, \ldots, n$). There is no loss of generality in assuming that the number of workers always

TABLE 5.31 Assignment Model

		Jobs				
		1	2	...	n	
Worker	1	c_{11}	c_{12}	...	c_{1n}	1
	2	c_{21}	c_{22}	...	c_{2n}	1
	⋮	⋮	⋮	⋮	⋮	⋮
	n	c_{n1}	c_{n2}	...	c_{nn}	1
		1	1	...	1	

equals the number of jobs, because we can always add fictitious workers or fictitious jobs to satisfy this assumption.

The assignment model is actually a special case of the transportation model in which the workers represent the sources, and the jobs represent the destinations. The supply (demand) amount at each source (destination) exactly equals 1. The cost of "transporting" worker i to job j is c_{ij}. In effect, the assignment model can be solved directly as a regular transportation model. Nevertheless, the fact that all the supply and demand amounts equal 1 has led to the development of a simple solution algorithm called the **Hungarian method.** Although the new solution method appears totally unrelated to the transportation model, the algorithm is actually rooted in the simplex method, just as the transportation model is.

5.4.1 The Hungarian Method[8]

We will use two examples to present the mechanics of the new algorithm. The next section provides a simplex-based explanation of the procedure.

Example 5.4-1

Joe Klyne's three children, John, Karen, and Terri, want to earn some money to take care of personal expenses during a school trip to the local zoo. Mr. Klyne has chosen three chores for his children: mowing the lawn, painting the garage door, and washing the family cars. To avoid anticipated sibling competition, he asks them to submit (secret) bids for what they feel is fair pay for each of the three chores. The understanding is that all three children will abide by their father's decision as to who gets which chore. Table 5.32 summarizes the bids received. Based on this information, how should Mr. Klyne assign the chores?

The assignment problem will be solved by the Hungarian method.

Step 1. For the original cost matrix, identify each row's minimum, and subtract it from all the entries of the row.

[8]As with the transportation model, the classical Hungarian method, designed primarily for *hand* computations, is something of the past and is presented here purely for historical reasons. Today, the need for such computational shortcuts is not warranted as the problem can be solved as a regular LP using highly efficient computer codes.

TABLE 5.32 Klyne's Assignment Problem

	Mow	Paint	Wash
John	\$15	\$10	\$9
Karen	\$9	\$15	\$10
Terri	\$10	\$12	\$8

Step 2. For the matrix resulting from step 1, identify each column's minimum, and subtract it from all the entries of the column.

Step 3. Identify the optimal solution as the feasible assignment associated with the zero elements of the matrix obtained in step 2.

Let p_i and q_j be the minimum costs associated with row i and column j as defined in steps 1 and 2, respectively. The row minimums of step 1 are computed from the original cost matrix as shown in Table 5.33.

Next, subtract the row minimum from each respective row to obtain the reduced matrix in Table 5.34.

The application of step 2 yields the column minimums in Table 5.34. Subtracting these values from the respective columns, we get the reduced matrix in Table 5.35.

TABLE 5.33 Step 1 of the Hungarian Method

	Mow	Paint	Wash	Row minimum
John	15	10	9	$p_1 = 9$
Karen	9	15	10	$p_2 = 9$
Terri	10	12	8	$p_3 = 8$

TABLE 5.34 Step 2 of the Hungarian Method

	Mow	Paint	Wash
John	6	1	0
Karen	0	6	1
Terri	2	4	0
Column minimum	$q_1 = 0$	$q_2 = 1$	$q_3 = 0$

TABLE 5.35 Step 3 of the Hungarian Method

	Mow	Paint	Wash
John	6	<u>0</u>	0
Karen	<u>0</u>	5	1
Terri	2	3	<u>0</u>

The cells with underscored zero entries provide the optimum solution. This means that John gets to paint the garage door, Karen gets to mow the lawn, and Terri gets to wash the family cars. The total cost to Mr. Klyne is $9 + 10 + 8 = \$27$. This amount also will always equal $(p_1 + p_2 + p_3) + (q_1 + q_2 + q_3) = (9 + 9 + 8) + (0 + 1 + 0) = \$27$. (A justification of this result is given in the next section.)

The given steps of the Hungarian method work well in the preceding example because the zero entries in the final matrix happen to produce a *feasible* assignment (in the sense that each child is assigned a distinct chore). In some cases, the zeros created by steps 1 and 2 may not yield a feasible solution directly, and further steps are needed to find the optimal (feasible) assignment. The following example demonstrates this situation.

Example 5.4-2

Suppose that the situation discussed in Example 5.4-1 is extended to four children and four chores. Table 5.36 summarizes the cost elements of the problem.

The application of steps 1 and 2 to the matrix in Table 5.36 (using $p_1 = 1, p_2 = 7, p_3 = 4, p_4 = 5, q_1 = 0, q_2 = 0, q_3 = 3$, and $q_4 = 0$) yields the reduced matrix in Table 5.37 (verify!).

The locations of the zero entries do not allow assigning unique chores to all the children. For example, if we assign child 1 to chore 1, then column 1 will be eliminated, and child 3 will not have a zero entry in the remaining three columns. This obstacle can be accounted for by adding the following step to the procedure outlined in Example 5.4-1:

Step 2a. If no feasible assignment (with all zero entries) can be secured from steps 1 and 2,

(i) Draw the *minimum* number of horizontal and vertical lines in the last reduced matrix that will cover *all* the zero entries.

TABLE 5.36 Assignment Model

		Chore			
		1	2	3	4
Child	1	\$1	\$4	\$6	\$3
	2	\$9	\$7	\$10	\$9
	3	\$4	\$5	\$11	\$7
	4	\$8	\$7	\$8	\$5

TABLE 5.37 Reduced Assignment Matrix

		Chore			
		1	2	3	4
Child	1	0	3	2	2
	2	2	0	0	2
	3	0	1	4	3
	4	3	2	0	0

TABLE 5.38 Application of Step 2a

		Chore 1	2	3	4
Child	1	0	3	2	2
	2	2	0	0	2
	3	0	*1*	4	3
	4	3	2	0	0

TABLE 5.39 Optimal Assignment

		Chore 1	2	3	4
Child	1	<u>0</u>	2	1	1
	2	3	0	<u>0</u>	2
	3	0	<u>0</u>	3	2
	4	4	2	0	<u>0</u>

(ii) Select the *smallest uncovered* entry, subtract it from every uncovered entry, then add it to every entry at the intersection of two lines.

(iii) If no feasible assignment can be found among the resulting zero entries, repeat step 2a. Otherwise, go to step 3 to determine the optimal assignment.

The application of step 2a to the last matrix produces the shaded cells in Table 5.38. The smallest unshaded entry (shown in italics) equals 1. This entry is added to the bold intersection cells and subtracted from the remaining shaded cells to produce the matrix in Table 5.39.

The optimum solution (shown by the underscored zeros) calls for assigning child 1 to chore 1, child 2 to chore 3, child 3 to chore 2, and child 4 to chore 4. The associated optimal cost is $1 + 10 + 5 + 5 = \$21$. The same cost is also determined by summing the p_i's, the q_j's, and the entry that was subtracted after the shaded cells were determined—that is, $(1 + 7 + 4 + 5) + (0 + 0 + 3 + 0) + (1) = \$21$.

AMPL Moment.

File amplEx5.4-2.txt provides the AMPL model for the assignment model. The model is very similar to that of the transportation model.

PROBLEM SET 5.4A

1. Solve the assignment models in Table 5.40.
   - **(a)** Solve by the Hungarian method.
   - **(b)** *TORA Experiment.* Express the problem as an LP and solve it with TORA.
   - **(c)** *TORA Experiment.* Use TORA to solve the problem as a transportation model.

TABLE 5.40 Data for Problem 1

(i)					(ii)				
\$3	\$8	\$2	\$10	\$3	\$3	\$9	\$2	\$3	\$7
\$8	\$7	\$2	\$9	\$7	\$6	\$1	\$5	\$6	\$6
\$6	\$4	\$2	\$7	\$5	\$9	\$4	\$7	\$10	\$3
\$8	\$4	\$2	\$3	\$5	\$2	\$5	\$4	\$2	\$1
\$9	\$10	\$6	\$9	\$10	\$9	\$6	\$2	\$4	\$5

(d) *Solver Experiment.* Modify Excel file solverEx5.3-1.xls to solve the problem.

(e) *AMPL Experiment.* Modify amplEx5.3-1b.txt to solve the problem.

2. JoShop needs to assign 4 jobs to 4 workers. The cost of performing a job is a function of the skills of the workers. Table 5.41 summarizes the cost of the assignments. Worker 1 cannot do job 3 and worker 3 cannot do job 4. Determine the optimal assignment using the Hungarian method.
3. In the JoShop model of Problem 2, suppose that an additional (fifth) worker becomes available for performing the four jobs at the respective costs of \$60, \$45, \$30, and \$80. Is it economical to replace one of the current four workers with the new one?
4. In the model of Problem 2, suppose that JoShop has just received a fifth job and that the respective costs of performing it by the four current workers are \$20, \$10, \$20, and \$80. Should the new job take priority over any of the four jobs JoShop already has?

*5. A business executive must make the four round trips listed in Table 5.42 between the head office in Dallas and a branch office in Atlanta.

The price of a round-trip ticket from Dallas is \$400. A discount of 25% is granted if the dates of arrival and departure of a ticket span a weekend (Saturday and Sunday). If the stay in Atlanta lasts more than 21 days, the discount is increased to 30%. A one-way

TABLE 5.41 Data for Problem 2

		Job 1	Job 2	Job 3	Job 4
Worker	1	\$50	\$50	—	\$20
	2	\$70	\$40	\$20	\$30
	3	\$90	\$30	\$50	—
	4	\$70	\$20	\$60	\$70

TABLE 5.42 Data for Problem 5

Departure date from Dallas	Return date to Dallas
Monday, June 3	Friday, June 7
Monday, June 10	Wednesday, June 12
Monday, June 17	Friday, June 21
Tuesday, June 25	Friday, June 28

ticket between Dallas and Atlanta (either direction) costs $250. How should the executive purchase the tickets?

***6.** Figure 5.6 gives a schematic layout of a machine shop with its existing work centers designated by squares 1, 2, 3, and 4. Four new work centers, I, II, III, and IV, are to be added to the shop at the locations designated by circles *a*, *b*, *c*, and *d*. The objective is to assign the new centers to the proposed locations to minimize the total materials handling traffic between the existing centers and the proposed ones. Table 5.43 summarizes the frequency of trips between the new centers and the old ones. Materials handling equipment travels along the rectangular aisles intersecting at the locations of the centers. For example, the one-way travel distance (in meters) between center 1 and location *b* is 30 + 20 = 50 m.

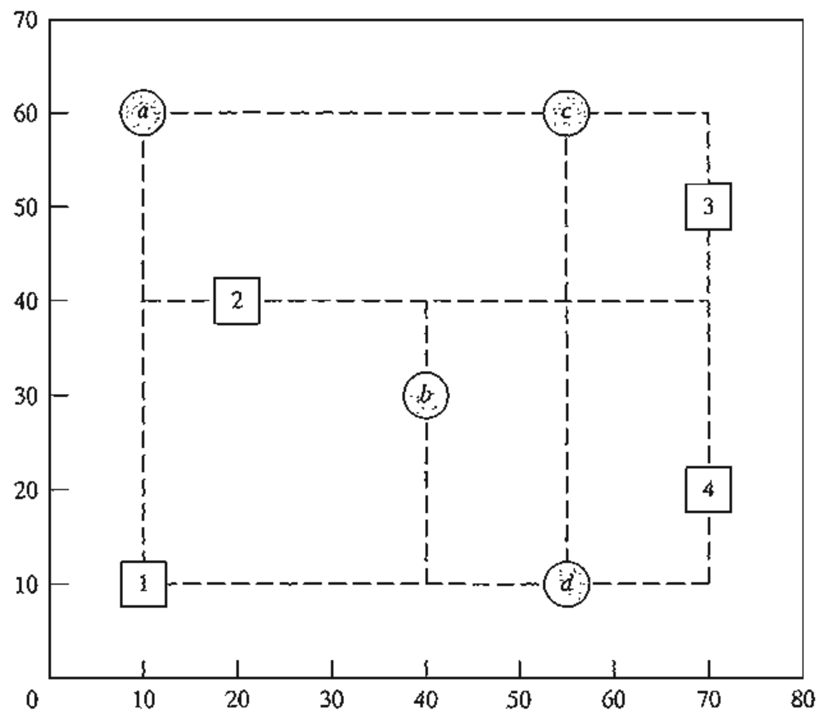


FIGURE 5.6

Machine shop layout for Problem 6, Set 5.4a

TABLE 5.43 Data for Problem 6

		New center			
		I	II	III	IV
	1	10	2	4	3
Existing	2	7	1	9	5
center	3	0	8	6	2
	4	11	4	0	7

7. In the Industrial Engineering Department at the University of Arkansas, INEG 4904 is a capstone design course intended to allow teams of students to apply the knowledge and skills learned in the undergraduate curriculum to a practical problem. The members of each team select a project manager, identify an appropriate scope for their project, write and present a proposal, perform necessary tasks for meeting the project objectives, and write and present a final report. The course instructor identifies potential projects and provides appropriate information sheets for each, including contact at the sponsoring organization, project summary, and potential skills needed to complete the project. Each design team is required to submit a report justifying the selection of team members and the team manager. The report also provides a ranking for each project in order of preference, including justification regarding proper matching of the team's skills with the project objectives. In a specific semester, the following projects were identified: Boeing F-15, Boeing F-18, Boeing Simulation, Cargil, Cobb-Vantress, ConAgra, Cooper, DaySpring (layout), DaySpring (material handling), J.B. Hunt, Raytheon, Tyson South, Tyson East, Wal-Mart, and Yellow Transportation. The projects for Boeing and Raytheon require U.S. citizenship of all team members. Of the eleven design teams available for this semester, four do not meet this requirement.

   Devise a procedure for assigning projects to teams and justify the arguments you use to reach a decision.

5.4.2 Simplex Explanation of the Hungarian Method

The assignment problem in which n workers are assigned to n jobs can be represented as an LP model in the following manner: Let c_{ij} be the cost of assigning worker i to job j, and define

$$x_{ij} = \begin{cases} 1, & \text{if worker } i \text{ is assigned to job } j \\ 0, & \text{otherwise} \end{cases}$$

Then the LP model is given as

$$\text{Minimize } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, \ldots, n$$

$$x_{ij} = 0 \text{ or } 1$$

The optimal solution of the preceding LP model remains unchanged if a constant is added to or subtracted from any row or column of the cost matrix (c_{ij}). To prove this point, let p_i and q_j be constants subtracted from row i and column j. Thus, the cost element c_{ij} is changed to

$$c'_{ij} = c_{ij} - p_i - q_j$$

Now

$$\sum_i \sum_j c'_{ij}x_{ij} = \sum_i \sum_j (c_{ij} - p_i - q_j)x_{ij} = \sum_i \sum_j c_{ij}x_{ij} - \sum_i p_i\Big(\sum_j x_{ij}\Big) - \sum_j q_j\Big(\sum_i x_{ij}\Big)$$

$$= \sum_i \sum_j c_{ij}x_{ij} - \sum_i p_i(1) - \sum_j q_j(1)$$

$$= \sum_i \sum_j c_{ij}x_{ij} - \text{constant}$$

Because the new objective function differs from the original one by a constant, the optimum values of x_{ij} must be the same in both cases. The development thus shows that steps 1 and 2 of the Hungarian method, which call for subtracting p_i from row i and then subtracting q_j from column j, produce an equivalent assignment model. In this regard, if a feasible solution can be found among the zero entries of the cost matrix created by steps 1 and 2, then it must be optimum because the cost in the modified matrix cannot be less than zero.

If the created zero entries cannot yield a feasible solution (as Example 5.4-2 demonstrates), then step 2a (dealing with the covering of the zero entries) must be applied. The validity of this procedure is again rooted in the simplex method of linear programming and can be explained by duality theory (Chapter 4) and the complementary slackness theorem (Chapter 7). We will not present the details of the proof here because they are somewhat involved.

The reason $(p_1 + p_2 + \cdots + p_n) + (q_1 + q_2 + \cdots + q_n)$ gives the optimal objective value is that it represents the dual objective function of the assignment model. This result can be seen through comparison with the dual objective function of the transportation model given in Section 5.3.4. [See Bazaraa and Associates (1990, pp. 499–508) for the details.]

5.5 THE TRANSSHIPMENT MODEL

The transshipment model recognizes that it may be cheaper to ship through intermediate or *transient* nodes before reaching the final destination. This concept is more general than that of the regular transportation model, where direct shipments only are allowed between a source and a destination.

This section shows how a transshipment model can be converted to (and solved as) a regular transportation model using the idea of a **buffer**.

Example 5.5-1

Two automobile plants, *P*1 and *P*2, are linked to three dealers, *D*1, *D*2, and *D*3, by way of two transit centers, *T*1 and *T*2, according to the network shown in Figure 5.7. The supply amounts at plants *P*1 and *P*2 are 1000 and 1200 cars, and the demand amounts at dealers *D*1, *D*2, and *D*3, are 800, 900, and 500 cars. The shipping costs per car (in hundreds of dollars) between pairs of nodes are shown on the connecting links (or arcs) of the network.

Transshipment occurs in the network in Figure 5.7 because the entire supply amount of 2200 (= 1000 + 1200) cars at nodes *P*1 and *P*2 could conceivably pass through any node of the

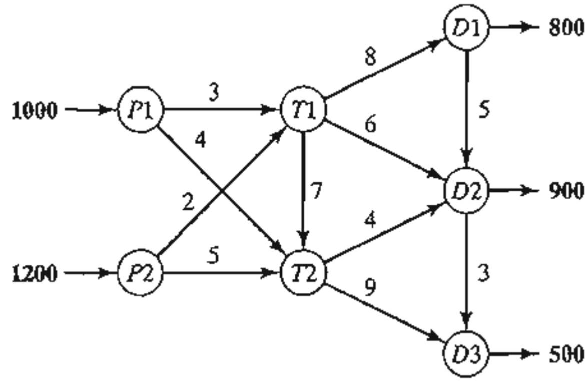


FIGURE 5.7
Transshipment network between plants and dealers

network before ultimately reaching their destinations at nodes $D1$, $D2$, and $D3$. In this regard, each node of the network with both input and output arcs ($T1$, $T2$, $D1$, and $D2$) acts as both a source and a destination and is referred to as a **transshipment node**. The remaining nodes are either **pure supply nodes** ($P1$ and $P2$) or **pure demand nodes** ($D3$).

The transshipment model can be converted into a regular transportation model with six sources ($P1, P2, T1, T2, D1,$ and $D2$) and five destinations ($T1, T2, D1, D2,$ and $D3$). The amounts of supply and demand at the different nodes are computed as

$$\begin{aligned}\text{Supply at a } \textit{pure supply node} &= \text{Original supply}\\ \text{Demand at a } \textit{pure demand node} &= \text{Original demand}\\ \text{Supply at a } \textit{transshipment node} &= \text{Original supply} + \text{Buffer amount}\\ \text{Demand at a } \textit{transshipment node} &= \text{Original demand} + \text{Buffer amount}\end{aligned}$$

The buffer amount should be sufficiently large to allow all of the *original* supply (or demand) units to pass through any of the *transshipment* nodes. Let B be the desired buffer amount; then

$$\begin{aligned}B &= \text{Total supply (or demand)}\\ &= 1000 + 1200 \text{ (or } 800 + 900 + 500)\\ &= 2200 \text{ cars}\end{aligned}$$

Using the buffer B and the unit shipping costs given in the network, we construct the equivalent regular transportation model as in Table 5.44.

The solution of the resulting transportation model (determined by TORA) is shown in Figure 5.8. Note the effect of transshipment: Dealer $D2$ receives 1400 cars, keeps 900 cars to satisfy its demand, and sends the remaining 500 cars to dealer $D3$.

PROBLEM SET 5.5A[9]

1. The network in Figure 5.9 gives the shipping routes from nodes 1 and 2 to nodes 5 and 6 by way of nodes 3 and 4. The unit shipping costs are shown on the respective arcs.
   (a) Develop the corresponding transshipment model.
   (b) Solve the problem, and show how the shipments are routed from the sources to the destinations.

[9]You are encouraged to use TORA, Excel Solver, or AMPL to solve the problems in this set.

TABLE 5.44 Transshipment Model

	T1	*T2*	*D1*	*D2*	*D3*	
*P*1	3	4	*M*	*M*	*M*	1000
*P*2	2	5	*M*	*M*	*M*	1200
*T*1	0	7	8	6	*M*	*B*
*T*2	*M*	0	*M*	4	9	*B*
*D*1	*M*	*M*	0	5	*M*	*B*
*D*2	*M*	*M*	*M*	0	3	*B*
	B	*B*	800 + *B*	900 + *B*	500	

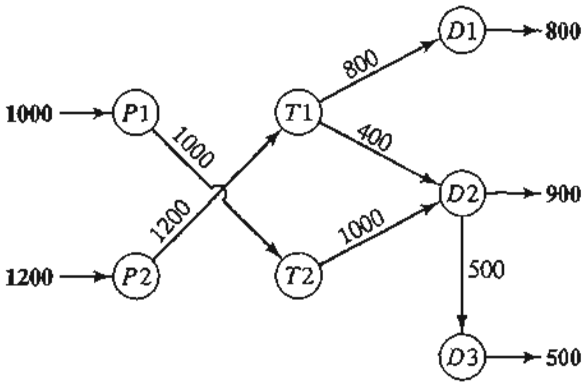


FIGURE 5.8
Solution of the transshipment model

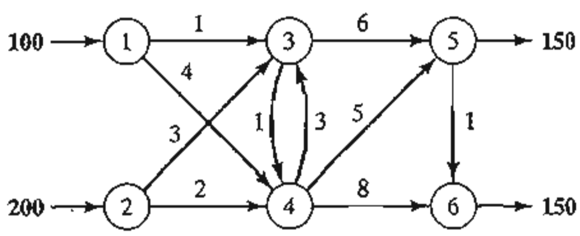


FIGURE 5.9
Network for Problem 1, Set 5.5a

2. In Problem 1, suppose that source node 1 can be linked to source node 2 with a unit shipping cost of $1. The unit shipping cost from node 1 to node 3 is increased to $5. Formulate the problem as a transshipment model, and find the optimum shipping schedule.
3. The network in Figure 5.10 shows the routes for shipping cars from three plants (nodes 1, 2, and 3) to three dealers (nodes 6 to 8) by way of two distribution centers (nodes 4 and 5). The shipping costs per car (in $100) are shown on the arcs.
   (a) Solve the problem as a transshipment model.
   (b) Find the new optimum solution assuming that distribution center 4 can sell 240 cars directly to customers.

*4. Consider the transportation problem in which two factories supply three stores with a commodity. The numbers of supply units available at sources 1 and 2 are 200 and 300; those demanded at stores 1, 2, and 3 are 100, 200, and 50, respectively. Units may be transshipped among the factories and the stores before reaching their final destination. Find the optimal shipping schedule based on the unit costs in Table 5.45.

5. Consider the oil pipeline network shown in Figure 5.11. The different nodes represent pumping and receiving stations. Distances in miles between the stations are shown on the network. The transportation cost per gallon between two nodes is directly proportional to the length of the pipeline. Develop the associated transshipment model, and find the optimum solution.
6. *Shortest-Route Problem.* Find the shortest route between nodes 1 and 7 of the network in Figure 5.12 by formulating the problem as a transshipment model. The distances between the different nodes are shown on the network. (*Hint:* Assume that node 1 has a net supply of 1 unit, and node 7 has a net demand also of 1 unit.)

FIGURE 5.10
Network for Problem 3, Set 5.5a

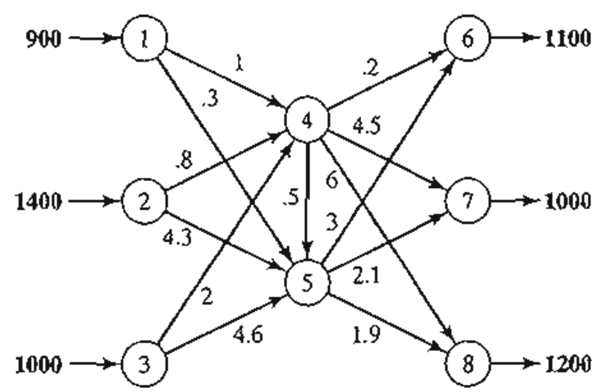


TABLE 5.45 Data for Problem 4

		Factory 1	Factory 2	Store 1	Store 2	Store 3
Factory	1	$0	$6	$7	$8	$9
	2	$6	$0	$5	$4	$3
Store	1	$7	$2	$0	$5	$1
	2	$1	$5	$1	$0	$4
	3	$8	$9	$7	$6	$0

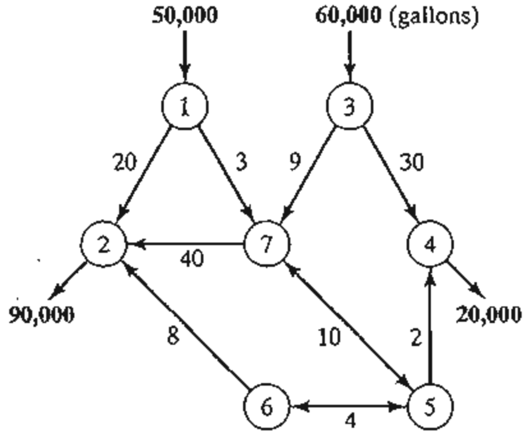


FIGURE 5.11
Network for Problem 5, Set 5.5a

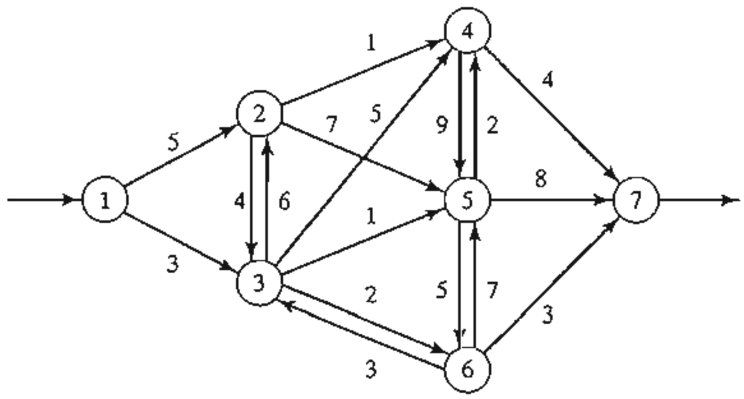


FIGURE 5.12
Network for Problem 6, Set 5.5a

7. In the transshipment model of Example 5.5-1, define x_{ij} as the amount shipped from node i to node j. The problem can be formulated as a linear program in which each node produces a constraint equation. Develop the linear program, and show that the resulting formulation has the characteristic that the constraint coefficients, a_{ij}, of the variable x_{ij} are

$$a_{ij} = \begin{cases} 1, \text{ in constraint } i \\ -1, \text{ in constraint } j \\ 0, \text{ otherwise} \end{cases}$$

8. An employment agency must provide the following laborers over the next 5 months:

Month	1	2	3	4	5
No. of laborers	100	120	80	170	50

Because the cost of labor depends on the length of employment, it may be more economical to keep more laborers than needed during some months of the 5-month planning horizon. The following table estimates the labor cost as a function of the length of employment:

Months of employment	1	2	3	4	5
Cost per laborer ($)	100	130	180	220	250

Formulate the problem as a linear program. Then, using proper algebraic manipulations of the constraint equations, show that the model can be converted to a transshipment model, and find the optimum solution. (*Hint*: Use the transshipment characteristic in Problem 7 to convert the constraints of the scheduling problem into those of the transshipment model.)

REFERENCES

Bazaraa, M., J. Jarvis, and H. Sherali, *Linear Programming and Network Flows*, 2nd ed., Wiley, New York, 1990.

Dantzig, G., *Linear Programming and Extensions*, Princeton University Press, Princeton, N.J., 1963.

Hansen, P. and R. Wendell, "A Note on Airline Commuting," *Interfaces*, Vol. 12, No. 1, pp. 85–87, 1982.

Murty, K., *Network Programming*, Prentice Hall, Upper Saddle River, NJ, 1992.